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HIGHER MATHEMATICS IN PROBLEMS AND EXERCISES

PART 1

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About the Book

In this two-volume textbook the authors tried to reveal the essence of the main notions and theorems encountered in courses of higher mathematics by presenting and solving specially selected problems and exercises.
Part 1 covers the following themes: analytic geometry on a plane and in space (with elements of vector algebra); fundamentals of linear algebra; introduction to analysis; differential calculus of functions of one or several variables; indefinite and definite integrals; and elements of linear programming.
Each section begins with a brief theoretical introduction. Typical problems are followed by detailed solutions.
The book is intended for engineering students.

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PART 1

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Contents

Preface to the English Edition

In this textbook the authors tried to reveal the essence of the main notions and theorems encountered in the course of higher matematics presently studied at higher schools by presenting and solving specially selected problems and exercises.

Each section is preceded by a short theoretical introduction contaning definitions and basic notions used in that section. The most difficult theoretical notions are developed (without proof) to achieve their better understanding by the students.

This two-volume bool has run into three editions in Russian and is widely used at higher schools in the USSR.

Specially for the English edition the authors prepared new material, adding some problems and including Chapter 10 into the second volume entitled "Fundamentals of Calculus of Variations".

The authors want to express their gratitude to Professor A. V. Efimov, D.Sc. (Phys.-Math.), the official reviewer, for his useful remarks and his beneficial attitude to their textbook in general.

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The Authors

Chapter 1

Plane Analytic Geometry

1.1. Rectangular and Polar Coordinates

1.1.1. Coordinates on a straight line. Division of a line segment in a given ratio. A point M on the coordinate x-axis with the abscissa x is customarily designated $M(x)$.

The distance d between the points $M_1(x_1)$ and $M_2(x_2)$ on the x-axis is specified by the formula

$$d = |x_2 - x_1| \tag{1}$$

whatever the positions of the points on the axis.

Suppose a closed interval $[AB]$ (A being the initial point of the line and B its end point) is given on an arbitrary straight line; then every third point C of that line divides the interval $[AB]$ in some ratio λ, where $\lambda = \pm |AC| : |CB|$. If the intervals $[AC]$ and $[CB]$ point in the same direction, then λ is given the plus sign, and if the intervals $[AC]$ and $[CB]$ point in opposite directions, then λ is given the minus sign. In other words, λ is positive if the point C lies between the points A and B, and negative if the point C lies on the straight line outside the interval $[AB]$.

If the points A and B lie on the x-axis, then the coordinate x of the point $C(x)$, dividing the interval between the points $A(x_1)$ and $B(x_2)$ in the λ ratio, can be determined by the formula

$$x = \frac{x_1 + \lambda x_2}{1 + \lambda}. \tag{2}$$

In particular, at $\lambda = 1$, the following formula is obtained for the coordinate of the midpoint of the closed interval:

$$x = \frac{x_1 + x_2}{2}. \tag{3}$$

1. Construct the points $A(3)$, $B(-2)$, $C(0)$, $D(\sqrt{2})$, $E(-3.5)$ on a straight line.

2. A closed interval $[AB]$ is divided by four points into five congruent parts. Determine the coordinate of the division point nearest to A if $A(-3)$, $B(7)$.

Solution. Suppose $C(x)$ is the desired point; then $\lambda = |AC| : |CB| = 1/4$. Consequently, we find from (2)

$$x = \frac{x_1 + \lambda x_2}{1 + \lambda} = \frac{-3 + (1/4)7}{1 + 1/4} = -1, \text{ i.e. } C(-1).$$

3. Given the end points $A(1)$, $B(5)$ of the closed interval $[AB]$; outside of this interval there is a point C whose distance to the point A is three times that to the point B. Determine the coordinate of the point C.

Solution. It is easy to see that $\lambda = -[AC] : [BC] = -3$ we recommend the reader to make a drawing). Thus we have

$$\bar{x} = \frac{1 - 3 \cdot 5}{1 - 3} = 7, \text{ i.e. } C(7).$$

4. Determine the distance between the points: (1) $M(3)$ and $N(-5)$; (2) $P(-11/2)$ and $Q(-5/2)$.

5. Find the coordinates of the midpoint of a closed interval if its end points are known: (1) $A(-6)$ and $B(7)$; (2) $C(-5)$ and $D(1/2)$.

6. Find the point M symmetric with respect to the point $N(-3)$ about the point $P(2)$.

7. Two points divide the closed interval $[AB]$ into three congruent parts. Determine the coordinates of the points of division if $A(-1)$, $B(5)$.

8. Given the points $A(-7)$, $B(-3)$. Outside the closed interval $[AB]$ there are points C and D, with $|CA| = |BD| = 0.5\,|AB|$. Determine the coordinates of the points C and D.

1.1.2. Rectangular coordinates on the plane. The simplest problems. If the rectangular Cartesian system of coordinates xOy is given on a plane, then a point M of that plane with the coordinates x and y is designated as $M(x; y)$.

The distance d between the points $M_1(x_1; y_1)$ and $M_2(x_2; y_2)$ is determined by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{1}$$

In particular, the distance d from the point $M(x; y)$ to the origin is determined by the formula

$$d = \sqrt{x^2 + y^2}. \tag{2}$$

The coordinates of the point $C(x; y)$ dividing the interval between the points $A(x_1; y_1)$ and $B(x_2; y_2)$ in a given ratio λ (see 1.1.1) are determined by the formulas

$$x = \frac{x_1 + \lambda x_2}{1 + \lambda}; \quad y = \frac{y_1 + \lambda y_2}{1 + \lambda}. \tag{3}$$

In particular, at $\lambda = 1$, formulas are obtained for the coordinates of the midpoint of the interval:

$$x = \frac{x_1 + x_2}{2}; \quad y = \frac{y_1 + y_2}{2}. \tag{4}$$

The area of the triangle with the vertices $A(x_1; y_1)$, $B(x_2; y_2)$, $C(x_3; y_3)$ is determined by the formula

$$S = \frac{1}{2} \left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right|$$
$$= \frac{1}{2} \left| (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1) \right|. \tag{5}$$

The formula specifying the area of the triangle can be written as

$$S = \frac{1}{2} |\Delta|, \tag{6}$$

where

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

(the concept of a third-order determinant is given in 1.5).

9. Construct points $A(4; 3)$, $B(-2; 5)$, $C(5; -2)$, $D(-4; -3)$, $E(-6; 0)$, $F(0; 4)$ on a coordinate plane.

10. Determine the distance between the points $A(3; 8)$ and $B(-5; 14)$.

Solution. Using formula (1), we obtain

$$d = \sqrt{(-5 - 3)^2 + (14 - 8)^2} = \sqrt{64 + 36} = 10.$$

11. Show that the triangle with the vertices $A(-3; -3)$, $B(-1; 3)$, $C(11; -1)$ is right-angled.

Solution. We find the lengths of the sides of the triangle:

$$|AB| = \sqrt{(-1 + 3)^2 + (3 + 3)^2} = \sqrt{40},$$

$$|BC| = \sqrt{(11 + 1)^2 + (-1 - 3)^2} = \sqrt{160},$$

$$|AC| = \sqrt{(11 + 3)^2 + (-1 + 3)^2} = \sqrt{200}.$$

Since $|AB|^2 = 40$, $|BC|^2 = 160$, $|AC|^2 = 200$, we have $|AB|^2 + |BC|^2 = |AC|^2$. The sum of the squares of the lengths of two sides of the triangle is equal to the square of the length of the third side. Hence we infer that the triangle ABC is right-angled and the side AC is its hypotenuse.

12. Given the end points $A(-2; 5)$, $B(4; 17)$ of the closed interval $[AB]$. There is a point C on that interval whose distance to A is twice that to B. Determine the coordinates of the point C.

Solution. Since $|AC| = 2|CB|$, we have $\lambda = |AC| : |CB| = 2$. Here $x_1 = -2$, $y_1 = 5$, $x_2 = 4$, $y_2 = 17$; consequently

$$\bar{x} = \frac{-2 + 2 \cdot 4}{1 + 2} = 2, \quad \bar{y} = \frac{5 + 2 \cdot 17}{1 + 2} = 13, \text{ i.e. } C(2; 13).$$

13. The point $C(2; 3)$ is the midpoint of the closed interval $[AB]$. Find the coordinates of the point A if $B(7; 5)$.

Solution. Here $\bar{x} = 2, \bar{y} = 3, x_2 = 7, y_2 = 5$, whence $2 = (x_1 + 7)/2, 3 = (y_1 + 5)/2$. Consequently $x_1 = -3, y_1 = 1$, i.e. $A(-3; 1)$.

14. Given the vertices $A(x_1; y_1), B(x_2; y_2), C(x_3; y_3)$ of the triangle ABC. Find the coordinates of the intersection point of the medians of the triangle.

Solution. Determining the coordinates of the point D, the midpoint of the closed interval $[AB]$, we obtain

$$x_D = (x_1 + x_2)/2, \quad y_D = (y_1 + y_2)/2.$$

A point M at which the medians intersect divides the interval $[CD]$ in the ratio $2:1$, reckoning from the point C. Consequently, the coordinates of the point M can be determined from the formulas

$$\bar{x} = \frac{x_3 + 2x_D}{1 + 2}, \quad \bar{y} = \frac{y_3 + 2y_D}{1 + 2},$$

that is

$$\bar{x} = \frac{x_3 + 2(x_1 + x_2)/2}{3}, \quad \bar{y} = \frac{y_3 + 2(y_1 + y_2)/2}{3}.$$

The final result is

$$\bar{x} = \frac{x_1 + x_2 + x_3}{3}, \quad \bar{y} = \frac{y_1 + y_2 + y_3}{3}.$$

15. Determine the area of the triangle with vertices $A(-2; -4)$, $B(2; 8)$ and $C(10; 2)$.

Solution. Using formula (5), we obtain

$$S = \frac{1}{2} \cdot |(2 + 2)(2 + 4) - (10 + 2)(8 + 4)| = \frac{1}{2}|24 - 144| = 60 \text{ (sq. units)}.$$

16. Determine the distance between the points: (1) $A(2; 3)$ and $B(-10; -2)$; (2) $C(\sqrt{2}; -\sqrt{7})$ and $D(2\sqrt{2}; 0)$.

17. Show that the triangle with vertices $A(4; 3)$, $B(7; 6)$ and $C(2; 11)$ is right-angled.

18. Show that the triangle with vertices $A(2; -1)$, $B(4; 2)$ and $C(5; 1)$ is an isosceles one.

19. Given the vertices of a triangle: $A(-1; -1)$, $B(0; -6)$ and $C(-10; -2)$. Find the length of the median drawn from the vertex A.

20. Given the end points $A(-3; 7)$ and $B(5; 11)$ of the closed interval $[AB]$. Three points divide this interval into four congruent parts. Determine the coordinates of the division points.

21. Find the area of the triangle with the vertices $A(1; 5)$, $B(2; 7)$, $C(4; 11)$.

22. Given three consecutive vertices of a parallelogram: $A(11; 4)$, $B(-1; -1)$, $C(5; 7)$. Determine the coordinates of the fourth vertex.

23. Given two vertices $A(3; 8)$ and $B(10; 2)$ of a triangle and the point of intersection $M(1; 1)$ of its medians. Find the coordinates of the third vertex of the triangle.

24. Given the vertices of a triangle: $A(7; 2)$, $B(1; 9)$ and $C(-8; -11)$. Find the distances between the point of intersection of the medians and the vertices of the triangle.

25. The points $L(0; 0)$, $M(3; 0)$ and $N(0; 4)$ are the midpoints of the sides of a triangle. Calculate the area of the triangle.

1.1.3. Polar coordinates. In the polar system of coordinates, the position of a point M on the plane is determined by its distance $|OM| = \rho$ from the pole O (ρ being the *polar radius vector* of the point) and by the angle θ formed by the closed interval $[OM]$ with the polar axis Ox (θ being the *polar angle* of the point). The angle θ is considered to be positive if it is reckoned from the polar axis counterclockwise.

If the point M has the polar coordinates $\rho > 0$ and $\theta > 0$, where $0 \leqslant \theta < 2\pi$, then it is also associated with an infinite number of pairs of polar coordinates $(\rho; \theta + 2k\pi)$ and $[-\rho; \theta + (2k + 1)\pi]$, where $k \in Z$.

If we make the rectangular Cartesian system of coordinates originate at the pole and direct the Ox axis along the polar axis, then the rectangular coordinates x and y of the point M and its polar coordinates ρ and θ will be connected by the following formulas:

$$x = \rho\cos\theta, \quad y = \rho\sin\theta; \tag{1}$$

$$\rho = \sqrt{x^2 + y^2}, \quad \tan\theta = y/x. \tag{2}$$

26. Construct points given by the polar coordinates $A(4; \pi/4)$, $B(2; 4\pi/3)$, $C(3; -\pi/6)$, $D(-3; \pi/3)$, $E(0; \alpha)$, $F(-1; -3\pi/4)$.

27. Find the polar coordinates of the point $M(1; -\sqrt{3})$, if the pole coincides with the origin and the polar axis coincides with the positive direction of the abscissa axis.

Solution. Proceeding from equalities (2), we find

$$\rho = \sqrt{1^2 + (-\sqrt{3})^2} = 2; \quad \tan\theta = -\sqrt{3}.$$

The point M evidently lies in the fourth quadrant and, consequently, $\theta = 5\pi/3$. Thus, $M(2; 5\pi/3)$.

28. Find the rectangular coordinates of the point $A(2\sqrt{2}; 3\pi/4)$, if the pole coincides with the origin and the polar axis is directed along the abscissa axis.

Solution. Using formulas (1), we obtain

$$x = 2\sqrt{2}\cos(3\pi/4) = -2, \quad y = 2\sqrt{2}\sin(3\pi/4) = 2.$$

Thus, $A(-2; 2)$.

29. Find the polar coordinates of the points $A(2\sqrt{3}; 2)$, $B(0; -3)$, $C(-4; 4)$, $D(\sqrt{2}; -\sqrt{2})$, $E(-\sqrt{2}; -\sqrt{6})$, $F(-7; 0)$.

30. Find the rectangular coordinates of the points $A(10; \pi/2)$, $B(2; 5\pi/4)$, $C(0; \pi/10)$, $D(1; -\pi/4)$, $E(-1; \pi/4)$, $F(-1; -\pi/4)$.

31. Determine the distance between the points $M_1(\rho_1; \theta_1)$ and $M_2(\rho_2; \theta_2)$.

Hint. Apply the law of cosines to the triangle OM_1M_2.

32. Determine the distance between the points $M(3;\ \pi/4)$ and $N(4;\ 3\pi/4)$.

33. Find the polar coordinates of a point symmetric with respect to the point $M(\rho;\ \theta)$ about the polar axis.

34. Find the polar coordinates of a point symmetric with respect to the point $M(\rho;\ \theta)$ about the pole.

35. Find the polar coordinates of points symmetric with respect to the points $(3;\ \pi/6)$, $(5;\ 2\pi/3)$ and $(2;\ -\pi/6)$: (1) about the pole; (2) about the polar axis.

36. Find the polar coordinates of a point symmetric with respect to the point $M(\rho;\ \theta)$ about the straight line passing through the pole at right angles to the polar axis.

1.1.4. Equation of a curve. Every curve on the xOy plane, considered as a set of points, is associated with the equation connecting the coordinates of any point $M(x;\ y)$ ("current point") lying on that curve. Such an equation is known as the *equation of a given curve.*

If the coordinates of any point lying on a curve are substituted into the equation of that curve, the equation becomes an identity. If, now, the coordinates of any point not lying on a curve are substituted into the equation of that curve, then the equation cannot be satisfied.

37. One end point of a closed interval is displaced along the abscissa axis and the other along the axis of ordinates. Find the equation of a curve described by the mid-point of that interval if the length of the interval is c.

Solution. Suppose $M(x;\ y)$ is the midpoint of the interval. The length of the interval $[OM]$ (the length of the median) is equal to half the hypotenuse, i.e. $|OM| = c/2$. On the other hand, $|OM| = \sqrt{x^2 + y^2}$ (the distance between the point M and the origin).

Thus we arrive at an equation

$$\sqrt{x^2 + y^2} = c/2, \quad \text{or} \quad x^2 + y^2 = c^2/4.$$

This is precisely the equation of the desired curve. In geometric interpretation this curve is a circle of radius $c/2$ with centre at the origin.

38. Set up the equation of a curve whose every point is at the same distance from the point $F(0;\ 1/4)$ as from the straight line $y = -1/4$.

Solution. Take an arbitrary point $M(x;\ y)$ on the sought-for curve. The distance between the point M and the point F can be determined by the formula specifying the distance between two points:

$$|MF| = \sqrt{(x-0)^2 + \left(y - \frac{1}{4}\right)^2}.$$

The distance between the point M and the straight line $y = -1/4$ is geometrically obvious (Fig. 1):

$$|MN| = |MK| + |KN| = y + \frac{1}{4}.$$

Since by the hypothesis the equality $|MF| = |MN|$ is satisfied for any point M

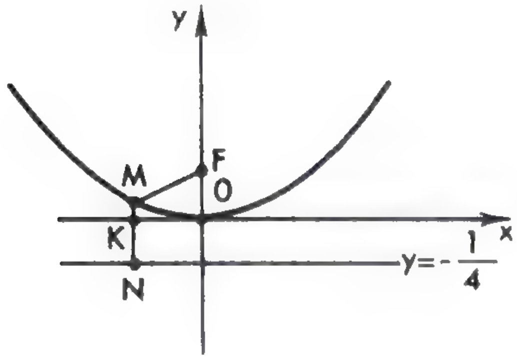

Fig. 1

of the sought-for curve, the equation of that curve can be written as

$$\sqrt{x^2+\left(y-\frac{1}{4}\right)^2}=y+\frac{1}{4}, \quad \text{or} \quad x^2+y^2-\frac{1}{2}y+\frac{1}{16}=y^2+\frac{1}{2}y+\frac{1}{16},$$

i.e. $y = x^2$.

The curve determined by the equation $y = x^2$ is called a *parabola*.

39. Set up the equation of the set of points the product of whose distances from the points $F_1(a; 0)$ and $F_2(-a; 0)$ is a constant quantity equal to a^2.

Solution. Take an arbitrary point $M(x; y)$ on the sought-for curve. Its distances from the points $F_1(a_1; 0)$ and $F_2(-a; 0)$ are $r_1 = \sqrt{(x-a)^2+y^2}$, $r_2 = \sqrt{(x+a)^2+y^2}$. It follows from the condition of the problem that $r_1r_2 = a^2$. Thus the desired curve is determined by the equation

$$\sqrt{(x-a)^2+y^2}\cdot\sqrt{(x+a)^2+y^2}=a^2.$$

Reduce this equation to the rational form:

$$(x^2+a^2+y^2-2ax)(x^2+a^2+y^2+2ax)=a^4,$$

that is

$$(x^2+a^2+y^2)^2-4a^2x^2=a^4,$$

or, finally,

$$(x^2+y^2)^2=2a^2(x^2-y^2).$$

The curve we have found is called a *lemniscate*.

40. Set up the equation of a lemniscate in polar coordinates and construct the curve.

Solution. In the equation $(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$ (see the preceding problem) we pass to polar coordinates, using the formulas $x = \rho\cos\theta$, $y = \rho\sin\theta$, and obtain

$(\rho^2\cos^2\theta + \rho^2\sin^2\theta)^2 = 2a^2(\rho^2\cos^2\theta - \rho^2\sin^2\theta)$, or $\rho^2 = 2a^2\cos 2\theta$.

This is an equation of the lemniscate in polar coordinates.

Let us now construct the curve. Solving the equation with respect to ρ, we find $\rho = \pm a\sqrt{2\cos 2\theta}$. We infer from the presence of the double sign "±" on the right-hand side of the equality and from the fact that the equation remains unchanged

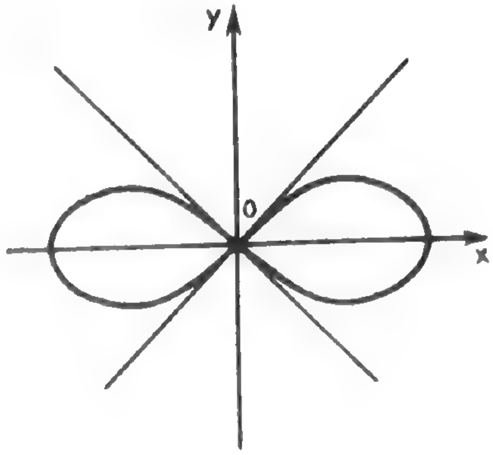

Fig. 2

upon the replacement of θ by $-\theta$, that the lemniscate is located symmetrically about the Ox and Oy axes. Let us investigate the shape of the lemniscate for the first quadrant, that is, for the case $\rho \geqslant 0$, $0 \leqslant \theta < \pi/2$. For these values of ρ and θ we have $\rho = a\sqrt{2} \cdot \sqrt{\cos 2\theta}$. It is easy to see that θ can vary only in the interval between 0 and $\pi/4$. Thus we see that the pertinent part of the curve is contained between the polar axis and the ray $\theta = \pi/4$. If $\theta = 0$, then $\rho = a\sqrt{2}$. With an increase in θ from 0 to $\pi/4$, the quantity ρ decreases to the value $\rho = 0$.

Taking the symmetry considerations into account, we can construct the lemniscate (Fig. 2).

41. Set up the equation of the set of points equidistant from the points $A(1; 1)$ and $B(3; 3)$.

Solution. Suppose the point M belongs to the sought-for set; then $|MA| = |MB|$. Using the formula specifying the distance between two points, we find

$$|MA| = \sqrt{(x-1)^2 + (y-1)^2},$$
$$|MB| = \sqrt{(x-3)^2 + (y-3)^2}$$

and the equation of the curve can be written in the form

$$\sqrt{(x-1)^2 + (y-1)^2} = \sqrt{(x-3)^2 + (y-3)^2}.$$

Squaring both sides of the last equality, we obtain

$$x^2 - 2x + 1 + y^2 - 2y + 1 = x^2 - 6x + 9 + y^2 - 6y + 9,$$

whence, after collecting like terms, we finally arrive at an equation

$$x + y - 4 = 0.$$

Thus we see that the sought-for set is a straight line which, as is known, serves as a perpendicular erected from the midpoint of the interval $[AB]$.

42. A point M is moving uniformly along a ray rotating uniformly about a pole. Set up the equation of the curve described by the point M if at the initial moment the rotating ray coincides with the polar axis and the point M with the pole; upon rotation of the ray through the angle $\theta = 1$ (one radian) the point M recedes from the pole by the distance a.

Solution. Since at the initial moment the quantities ρ and θ are equal to zero, and then both of them increase in proportion to time, it is easy to see that they are con-

nected by a direct proportional relation $\rho/\theta = \text{const}$. But $\rho = a$ at $\theta = 1$; consequently, $\rho/\theta = a/1$, i.e. $\rho = a\theta$. The curve $\rho = a\theta$ is known as the *spiral of Archimedes*.

43. A circle of diameter a rolls without slipping along the outer circumference of another circle of the same diameter. Use polar coordinates to set up an equation of the curve described by some fixed point of the rolling circle.

Solution. Shown in Fig. 3 are: C_1 — the initial position of the centre of the rolling circle; A — the initial position of the point describing the desired curve (the point A is diametrically opposite to the point B where the circles contact each other at the initial moment); C_2 — the centre of the stationary circle; C_3 — the centre of the rolling circle in its new position; M — the new position of the point A describing the desired curve. (After the displacement of the circle with centre C_1 to its new position with centre at C_3, the point P will be at Q. The point B will turn out at D, and, since the rolling is without slipping, we shall have $\widehat{BQ} = \widehat{DQ}$, $\widehat{QC_2B} = \widehat{QC_3D}$.)

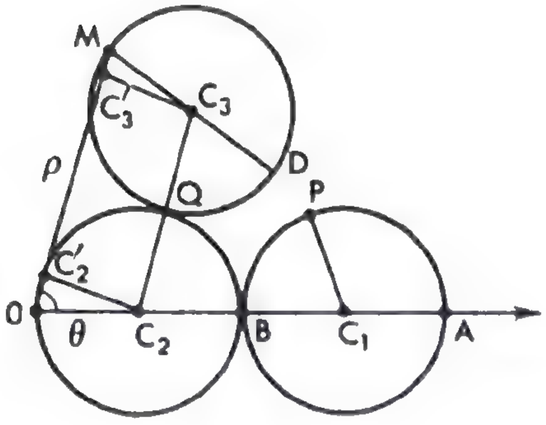

Fig. 3

The positions of the pole O and the polar axis Ox are shown in the figure. Set up the equation satisfied by the coordinates of any point $M(\rho; \theta)$ of the desired curve.

It is easy to see that $\widehat{MC_3Q} = \widehat{OC_2Q}$, and by virtue of this equality the rectangle OC_2C_3M is an isosceles trapezoid with the smaller base $|C_2C_3| = a$; C_2C_2' and C_3C_3' are perpendiculars dropped from the points C_2 and C_3 on the straight line OM. Hence we have

$$\rho = |OC_2'| + |C_2'C_3'| + |C_3'M| = \frac{a}{2}\cos\theta + a + \frac{a}{2}\cos\theta = a(1 + \cos\theta).$$

Thus, in polar coordinates the equation of the desired curve has the form $\rho = a(1 + \cos\theta)$; this curve is called a ***cardioid***.

Since the equation of the cardioid does not change upon a substitution of $-\theta$ for θ, the cardioid is symmetric about the polar axis. If θ increases from 0 to π, then ρ decreases from $2a$ to 0.

44. Find the equation of the set of points equidistant from the points $A(2; 0)$ and $B(0; 1)$.

45. What curve is determined by the equation $x = y$?

46. What curve is determined by the equation $x = -y$?

47. Set up the equation of the set of points the sum of the squares of whose distances from the points $A(2; 0)$ and $B(0; 2)$ is equal to the square of the distance between the points A and B.

48. Set up the equation of the set of points the sum of whose distances from the points $A(1; 0)$ and $B(0; 1)$ is equal to 2.

49. Set up the equation of the circle with centre at the pole in the polar system of coordinates.

50. Set up the equation of the half-line passing through the pole and forming an angle α with the polar axis in the polar system of coordinates.

51. Set up the equation of the circle of diameter a in polar coordinates if the pole lies on the circle and the polar axis passes through the centre of the circle.

1.1.5. Parametric equations of a curve. In the search for the equation of a set of points, it is sometimes more convenient to express the coordinates x and y of an arbitrary point of that set in terms of some auxiliary quantity t (called a *parameter*), that is, to consider the system of equations

$$x = \varphi(t), \quad y = \psi(t).$$

Such a representation of the desired curve is called *parametric* and the equations of the system, *the parametric equations of the given curve.*

The elimination of the parameter t from the system (if it is possible) leads to an equation connecting x and y, that is, to an ordinary equation of a curve of the form $f(x, y) = 0$.

52. Set up the parametric equations of a circle.

Solution. Let us consider a circle of radius a with centre at the origin (Fig. 4).

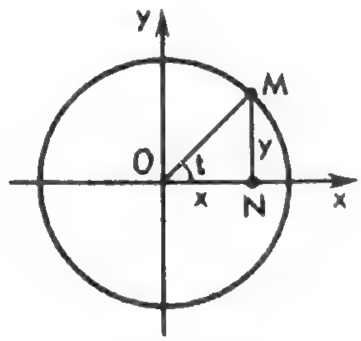

Fig. 4

Take an arbitrary point $M(x; y)$ on that circle and assume that the angle formed by the radius OM and the abscissa axis is the parameter t. It follows from the triangle OMN that $x = a\cos t$, $y = a\sin t$. Thus the equations

$$x = a\cos t, \quad y = a\sin t$$

are the parametric equations of the circle.

Eliminating the parameter t from these equations, we obtain an ordinary equation of a circle. To eliminate the parameter in the present case it is sufficient to square each of the equations and add up the results:

$$\begin{array}{r} x^2 = a^2\cos^2 t \\ + \\ y^2 = a^2\sin^2 t \\ \hline x^2 + y^2 = a^2. \end{array}$$

The last equation is the equation of a circle of radius a with centre at the origin.

53. Set up the parametric equations of the curve described by a fixed point of a circle rolling without slipping along a stationary line.

Solution. Suppose a circle of radius a rolls without slipping to the right along a horizontal line (Fig. 5). Assume this line to be the x-axis and put the origin at some point O of the axis. Take as the fixed point of the circle (whose displacement originates the desired curve) the point which coincides with the point O when the circle is in the requisite position. The parameter t is the angle of rotation of the radius of the circle passing through the fixed point.

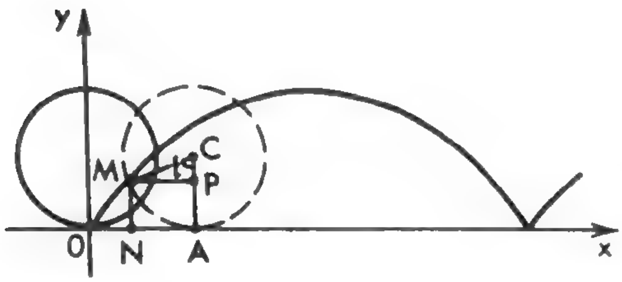

Fig. 5

Suppose that at a certain time moment the circle touches the axis at a point A. The fixed point of the circle will occupy the position $M(x; y)$ corresponding to the angle t of rotation of the radius CM ($t = \widehat{ACM}$). Since the rolling is without slipping, $|OA| = \overset{\frown}{MA} = at$. Using this equality, express the coordinates of the point M in terms of t:

$$x = |ON| = |OA| - |NA| = \overset{\frown}{MA} - |NA| = at - a\sin t = a(t - \sin t),$$

$$y = |NM| = |AP| = |AC| - |PC| = a - a\cos t = a(1 - \cos t).$$

It follows that the parametric equations of the desired curve have the form

$$x = a(t - \sin t), \quad y = a(1 - \cos t).$$

This curve is called a *cycloid*; it is represented in Fig. 5.

54. What curve is specified by the parametric equations $x = t^2, y = t^2$?

Solution. Elimination of the parameter t leads to an equation $y = x$. But by virtue of parametric equations, $x \geqslant 0$, $y \geqslant 0$. Consequently, the given parametric equations specify a ray, the bisector of the first quadrant.

55. What curve is specified by the parametric equations $x = \cos t, y = \cos^2 t$?

Solution. Substitution of x for $\cos t$ in the second equation leads to an equation of a parabola $y = x^2$. It follows from the parametric equations that $|x| \leqslant 1$, $0 \leqslant y \leqslant 1$. Thus the parametric equations specify the arc AOB of the parabola $y = x^2$, where $A(-1; 1)$, $B(1; 1)$.

56. What curve is specified by the equations $x = \sin t$, $y = \operatorname{cosec} t$?

Solution. Since $y = 1/\sin t$, we can eliminate t and obtain an equation $y = 1/x$, expressing an inversely proportional relationship between the quantities x and y. Taking into account that $|x| \leqslant 1$, $|y| \geqslant 1$, we infer that the curve specified by the parametric equations $x = \sin t$, $y = \operatorname{cosec} t$ has the shape shown in Fig. 6.

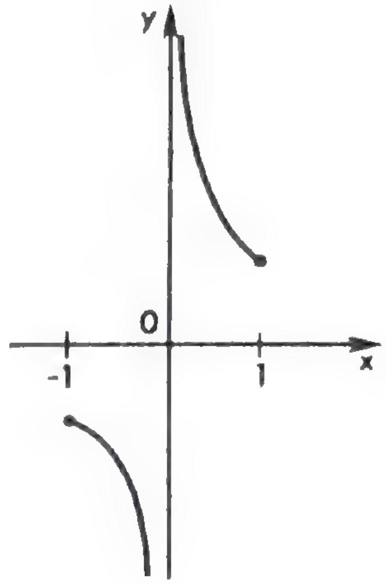

Fig. 6

57. What curve is specified by the equations $x = 2t$, $y = 4t$?

58. The curve is specified by the parametric equations $x = a\cos t$, $y = b\sin t$. Find its equation in the rectangular system of coordinates.

Hint. Divide the first equation by a and the second by b and then eliminate t.

59. The curve is specified by the parametric equations $x = a\sec t$, $y = b\tan t$. Find its equation in the rectangular system of coordinates.

60. What curve is determined by the equations $x = \cos^2 t$, $y = \sin^2 t$?

61. The curve specified by the parametric equations $x = a\cos^3 t$, $y = a\sin^3 t$ is called an *asteroid*. Eliminating t, find the equation of the asteroid in the rectangular system of coordinates.

62. A thread is wound clockwise round a circle described from the centre O by the radius a. Suppose one end of the thread is at a point $A(a; 0)$. Let us unwind the thread (counterclockwise), tightening it all the while by pulling it by the other end. Set up the parametric equations of the curve described by the end of the thread for the case when the parameter t is the angle between the radius OA and the radius OB drawn to the point of tangency of the circle with the tightened thread, the latter being in an arbitrary position.

1.2. A Straight Line

1.2.1. General equation of a straight line. Every first-degree equation with respect to x and y, that is, an equation of the form

$$Ax + By + C = 0$$

(where A, B and C are constant factors, with $A^2 + B^2 \neq 0$) specifies a certain line on a plane. This is the *general form of equation of a straight line.*

Special cases. 1. $C = 0$; $A \neq 0$; $B \neq 0$. The straight line specified by the equation $Ax + By = 0$ passes through the origin.

2. $A = 0$; $B \neq 0$; $C \neq 0$. The line specified by the equation $By + C = 0$ (or $y = b$, where $b = -C/B$) is parallel to the x-axis.

3. $B = 0$; $A \neq 0$; $C \neq 0$. The line specified by the equation $Ax + C = 0$ (or $x = a$, where $a = -C/A$) is parallel to the y-axis.

4. $B = C = 0$; $A \neq 0$. The line specified by the equation $Ax = 0$ (or $x = 0$, since $A \neq 0$) coincides with the y-axis.

5. $A = C = 0$; $B \neq 0$. The line specified by the equation $By = 0$ (or $y = 0$, since $B \neq 0$) coincides with the x-axis.

1.2.2. Slope-intercept form of equation of a straight line. If $B \neq 0$ in the general equation of a straight line, then, solving the equation with respect to y, we obtain an equation of the form

$$y = kx + b$$

(here $k = -A/B$, $b = -C/B$). This is the *slope-intercept form of equation of a straight line* since $k = \tan\alpha$, where α is the angle which the straight line makes with the positive direction of the x-axis. The constant term b of the equation is equal to the ordinate of the point of intersection of the straight line with the y-axis.

1.2.3. Intercept form of equation of a straight line. If $C \neq 0$ in the general equation of a straight line, then dividing all its terms by $-C$, we obtain an equation of the form

$$\frac{x}{a} + \frac{y}{b} = 1$$

(here $a = -C/A$, $b = -C/B$). This is the *intercept form of equation of a straight line,* a being the abscissa of the point of intersection of the line with the x-axis and b, the ordinate of the point of intersection of the line with the y-axis. Therefore, a and b are called the intercepts of a straight line on the coordinate axes, or the x and y intercepts.

1.2.4. Normal form of equation of a straight line. If both sides of the general equation of the line $Ax + By + C = 0$ are multiplied by the number $\mu = 1/(\pm\sqrt{A^2 + B^2})$ (which is known as a *normalization factor*), with the sign before the radical chosen so that the condition $\mu \cdot C < 0$ is satisfied, the following equation results:

$$x\cos\varphi + y\sin\varphi - p = 0.$$

This is the *normal form of equation of a straight line.* Here p is the length of the perpendicular let fall from the origin on the line, and φ is the angle formed by this perpendicular and the positive direction of the x-axis.

63. Set up the equation of a line intercepting a segment $b = -3$ on the ordinate axis and forming an angle $\alpha = \pi/6$ with the positive direction of the abscissa axis.

Solution. We first find the slope $k = \tan(\pi/6) = 1/\sqrt{3}$. Then, using the slope-intercept form of equation, we obtain $y = (1/\sqrt{3})x - 3$; deleting the denominator and transposing all the terms into the left-hand side, we obtain the general equation of a straight line $x - \sqrt{3}y - 3\sqrt{3} = 0$.

64. Set up the equation of a straight line intercepting line segments $a = 2/5$, $b = -1/10$ on the coordinate axes.

Solution. Using the intercept form of equation of a straight line, we obtain

$$\frac{x}{2/5} + \frac{y}{(-1/10)} = 1.$$

This equation can be rewritten in the form $(5/2)x - 10y = 1$, or $5x - 20y - 2 = 0$ (the general form of equation of a line).

65. Given the general equation of the line $12x - 5y - 65 = 0$. Write: (1) the slope-intercept form of equation; (2) the intercept form of equation; (3) the normal form of equation.

Solution. (1) Solving the equation of the line with respect to y, we obtain the slope-intercept form of the equation:

$$y = (12/5)x - 13.$$

Here $k = 12/5$, $b = -13$.

(2) Transposing the constant term of the equation into the right-hand side and dividing both sides by 65, we obtain $(12/65)x - (5/65)y = 1$. Imparting to the last equation the form

$$\frac{x}{65/12} + \frac{y}{(-65/5)} = 1,$$

we obtain the intercept form of equation of the given line. Here $a = 65/12$, $b = -65/5 = -13$.

(3) We find the normalization factor $\mu = 1/\sqrt{12^2 + (-5)^2} = 1/13$. Multiplying both sides of the general equation by this factor, we obtain the normal form of equation of the line

$$(12/13)x - (5/13)y - 5 = 0.$$

Here $\cos\varphi = 12/13$, $\sin\varphi = -5/13$, $p = 5$.

66. Construct the straight lines: (1) $x - 2y + 5 = 0$; (2) $2x + 3y = 0$; (3) $5x - 2 = 0$; (4) $2y + 7 = 0$.

Solution. (1) Putting $x = 0$ in the equation, we obtain $y = 5/2$. Consequently, the straight line intersects the ordinate axis at a point $B(0; 5/2)$. Assuming $y = 0$, we obtain $x = -5$, that is, the line intersects the abscissa axis at a point $A(-5; 0)$. It remains to draw a line through the points A and B (Fig. 7).

(2) The straight line $2x + 3y = 0$ passes through the origin since its equation contains no constant term. Now we assign some value to x in the equation of the line, say, $x = 3$ and get $6 + 3y = 0$, i.e. $y = -2$. We obtain a point $M(3; -2)$. It remains to draw a line through the origin and the point M.

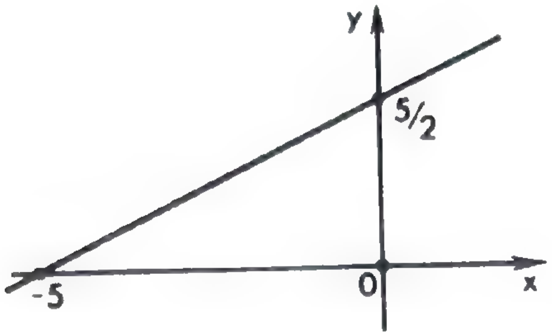

Fig. 7

(3) Solving the equation of the line with respect to x, we obtain $x = 2/5$. This line is parallel to the ordinate axis and intercepts a line segment equal to $2/5$ on the abscissa axis.

(4) In a similar way we obtain an equation $y = -7/2$; this line is parallel to the abscissa axis.

67. Given the equation of the straight line $(x + 2\sqrt{5})/4 + (y - 2\sqrt{5})/2 = 0$. Write: (1) the general equation of this line; (2) the slope-intercept form of this equation; (3) the intercept form of the equation; (4) the normal form of the equation.

68. What angle does the line $2x + 2y - 5 = 0$ make with the positive direction of the abscissa axis?

69. Determine the area of the triangle formed by the line $4x + 3y - 36 = 0$ and the coordinate axes.

70. Can the equation of the line $20x + 21y = 0$ be written in an intercept form?

71. Construct the straight lines: (1) $4x - 5y + 15 = 0$; (2) $2x - y = 0$; (3) $7x - 10 = 0$; (4) $2y + 3 = 0$.

72. Set up the equation of the straight line intercepting a line segment $b = 1$ on the ordinate axis and forming an angle $\alpha = 2\pi/3$ with the positive direction of the abscissa axis.

73. A line intercepts congruent positive segments on the coordinate axes. Set up the equation of the line if the area of the triangle formed by the line and the coordinate axes is equal to 8 sq. units.

74. Set up the equation of the line passing through the origin and the point $A(-2; -3)$.

75. Set up the equation of the line passing through the point $A(2; 5)$ and intercepting a segment $b = 7$ on the ordinate axis.

76. Set up the equations of the straight lines which pass through the point $M(-3; -4)$ and are parallel to the coordinate axes.

77. Set up the equation of the straight line intercepting congruent segments on the coordinate axes if the length of the line segment contained between the coordinate axes is equal to $5\sqrt{2}$.

1.2.5. An angle between straight lines. Two-point form of equation of a straight line. The *acute angle between the lines* $y = k_1x + b_1$ and $y = k_2x + b_2$ is determined by the formula

$$\tan\alpha = \left|\frac{k_2 - k_1}{1 + k_1k_2}\right|. \qquad (1)$$

The *condition of parallelism* of the lines has the form $k_1 = k_2$.

The *condition of perpendicularity* of the lines has the form $k_1 = -1/k_2$.

The *slope-k-intercept form of the equation of the line passing through the point* $M(x_1; y_1)$ is

$$y - y_1 = k(x - x_1). \qquad (2)$$

The *equation of the line passing through the points* $M_1(x_1; y_1)$ and $M_2(x_2; y_2)$ is written in the form

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}, \qquad (3)$$

and the slope of that line can be found from the formula

$$k = \frac{y_2 - y_1}{x_2 - x_1}. \qquad (4)$$

If $x_1 = x_2$, then the equation of the line passing through the points M_1 and M_2 has the form $x = x_1$.

If $y_1 = y_2$, then the equation of the line passing through the points M_1 and M_2 has the form $y = y_1$.

1.2.6. Intersection of lines. The distance from a point to a straight line. Pencil of lines. If $A_1/A_2 \neq B_1/B_2$, then the coordinates of the point of intersection of the lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ can be found by simultaneously solving the equations of these lines.

The *distance between the point* $M(x_0; y_0)$ *and the line* $Ax + By + C = 0$ can be found by the formula

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}. \qquad (1)$$

The bisectors of the angles between the lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ are determined by the equations

$$\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}} \pm \frac{A_2x + B_2y + C_2}{\sqrt{A_2^2 + B_2^2}} = 0. \qquad (2)$$

If the intersecting lines are specified by the equations $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$, then the equation

$$A_1x + B_1y + C_1 + \lambda(A_2x + B_2y + C_2) = 0, \qquad (3)$$

where λ is a numerical factor, specifies a straight line passing through the point of intersection of the given lines. Assigning various values to λ in the last equation, we obtain various lines belonging to a *pencil* of lines whose centre is the point of intersection of the given lines.

78. Determine the acute angle between the lines $y = -3x + 7$ and $y = 2x + 1$.

Solution. Putting $k_1 = -3$, $k_2 = 2$ in formula (1) in 1.2.5, we obtain

$$\tan\varphi = \left|\frac{2-(-3)}{1-(-3)\cdot 2}\right| = 1, \text{ i.e. } \varphi = \frac{\pi}{4}.$$

79. Show that the straight lines $4x - 6y + 7 = 0$ and $20x - 30y - 11 = 0$ are parallel.

Solution. Reducing the equation of each line to the slope-intercept form, we obtain

$$y = (2/3)x + 7/6 \text{ and } y = (2/3)x - 11/30.$$

The slopes of these lines are equal to $k_1 = k_2 = 2/3$, that is, the lines are parallel.

80. Show that the lines $3x - 5y + 7 = 0$ and $10x + 6y - 3 = 0$ are perpendicular.

Solution. Reducing the equations to the slope-intercept form, we obtain

$$y = (3/5)x + 7/5 \quad \text{and} \quad y = (-5/3)x + 1/2.$$

Here $k_1 = 3/5$, $k_2 = -5/3$. Since $k_1 = -1/k_2$, the lines are perpendicular.

81. Derive the equation of the straight line passing through the points $M(-1; 3)$ and $N(2; 5)$.

Solution. Putting $x_1 = -1, y_1 = 3, x_2 = 2, y_2 = 5$ in equation (3), 1.2.5, we obtain

$$\frac{y-3}{5-3} = \frac{x+1}{2+1}, \quad \text{or} \quad \frac{y-3}{2} = \frac{x+1}{3}.$$

Thus we see that the desired equation is of the form $2x - 3y + 11 = 0$.

It may be of use to verify that the equation has been correctly derived. For that purpose, it suffices to show that the coordinates of the points M and N satisfy the equation of a straight line. Indeed, the equalities $2(-1) - 3 \cdot 3 + 11 = 0$, $2 \cdot 2 - 3 \cdot 5 + 11 = 0$ are identically satisfied.

82. Derive the equation of the straight line passing through the points $A(-2; 4)$ and $B(-2; -1)$.

Solution. Since $x_1 = x_2 = -2$, the line satisfies the equation $x = -2$ (is parallel to the ordinate axis).

83. Show that the lines $3x - 2y + 1 = 0$ and $2x + 5y - 12 = 0$ intersect, and find the coordinates of the intersection point.

Solution. Since $3/2 \neq (-2)/5$, the lines intersect. Solving the system of equations

$$\begin{cases} 3x - 2y + 1 = 0, \\ 2x + 5y - 12 = 0, \end{cases}$$

we find $x = 1, y = 2$, that is, the lines intersect at the point $(1; 2)$.

84. Determine the distance from the point $M(x_0; y_0)$ to the line $Ax + By + C = 0$ without using the normal form of the equation of the line.

Solution. The problem reduces to determining the distance between the points $M(x_0; y_0)$ and N, where N is the foot of the perpendicular dropped from the point M onto the given line. Let us derive the equation of the line (MN). Since the slope of the given line is equal to $-A/B$, the slope of the line (MN) is equal to B/A(from

the condition of perpendicularity) and the equation of the latter is of the form $y - y_0 = (B/A)(x - x_0)$. This equation can be rewritten as $(x - x_0)/A = (y - y_0)/B$.

To determine the coordinates of the point N, we solve the system of equations

$$Ax + By + C = 0 \quad \text{and} \quad (x - x_0)/A = (y - y_0)/B.$$

Let us introduce an auxiliary unknown quantity t:

$$(x - x_0)/A = (y - y_0)/B = t.$$

Then $x = x_0 + At$, $y = y_0 + Bt$. Substituting these expressions into the equation of the given line, we obtain $A(x_0 + At) + B(y_0 + Bt) + C = 0$, whence

$$t = -(Ax_0 + By_0 + C)/(A^2 + B^2).$$

Substituting now the value of t into the equations $x = x_0 + At$ and $y = y_0 + Bt$, we determine the coordinates of the point N:

$$x = x_0 - A\frac{Ax_0 + By_0 + C}{A^2 + B^2}, \quad y = y_0 - B\frac{Ax_0 + By_0 + C}{A^2 + B^2}.$$

It remains to determine the distance between the points M and N:

$$d = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$
$$= \sqrt{\left(A\frac{Ax_0 + By_0 + C}{A^2 + B^2}\right)^2 + \left(B\frac{Ax_0 + By_0 + C}{A^2 + B^2}\right)^2} = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}.$$

85. Determine the distance from the point $M(1; 2)$ to the line $20x - 21y - 58 = 0$.

Solution. We have

$$d = \frac{|20 \cdot 1 - 21 \cdot 2 - 58|}{\sqrt{400 + 441}} = \frac{|20 - 42 - 58|}{29} = \frac{|-80|}{29} = 2\frac{22}{29}.$$

86. Given the line l: $4x - 3y - 7 = 0$. Which of the points $A(5/2; 1)$, $B(3; 2)$, $C(1; -1)$, $D(0; -2)$, $E(4; 3)$, $F(5; 2)$ lie on that line?

Solution. If a point lies on a line, then its coordinates must satisfy the equation of the line. We have: $A \in l$ since $4 \cdot 5/2 - 3 \cdot 1 - 7 = 0$; $B \notin l$ since $4 \cdot 3 - 3 \cdot 2 - 7 \neq 0$; $C \in l$ since $4 \cdot 1 - 3(-1) - 7 = 0$; $D \notin l$ since $4 \cdot 0 - 3(-2) - 7 \neq 0$; $E \in l$ since $4 \cdot 4 - 3 \cdot 3 - 7 = 0$; $F \notin l$ since $4 \cdot 5 - 3 \cdot 2 - 7 \neq 0$.

87. Derive the equation of the line which passes through the point $M(-2; -5)$ and is parallel to the line $3x + 4y + 2 = 0$.

Solution. Solving the last equation with respect to y, we obtain $y = -(3/4)x - 1/2$. Consequently, from the condition of parallelism, the slope of the desired straight line is equal to $-3/4$. Using equation (2) in 1.2.5, we obtain

$$y - (-5) = -\frac{3}{4}[x - (-2)], \quad \text{i.e.} \quad 3x + 4y + 26 = 0.$$

88. Given the vertices of a triangle: $A(2; 2)$, $B(-2; -8)$ and $C(-6; -2)$. Derive the equations of the medians of the triangle.

Solution. We find the coordinates of the midpoints of the sides BC, AC and AB:

$$x' = \frac{-2-6}{2} = -4, \quad y' = \frac{-8-2}{2} = -5; \quad A_1(-4; -5);$$

$$x'' = \frac{2-6}{2} = -2, \quad y'' = \frac{2-2}{2} = 0; \quad B_1(-2; 0);$$

$$x''' = \frac{2-2}{2} = 0, \quad y''' = \frac{2-8}{2} = -3; \quad C_1(0; -3).$$

The equations of the medians are found by using two-point form of equation of a line. The equation of the median AA_1 is

$$\frac{y-2}{-5-2} = \frac{x-2}{-4-2}, \quad \text{or} \quad \frac{y-2}{7} = \frac{x-2}{6}, \text{ i.e. } \quad 7x - 6y - 2 = 0.$$

Next we find the equation of the median BB_1. Since the points $B(-2; -8)$ and $B_1(-2; 0)$ have similar abscissas, the median BB_1 is parallel to the axis of ordinates. Its equation is $x + 2 = 0$.

The equation of the median CC_1 is

$$\frac{y+2}{-3+2} = \frac{x+6}{0+6}, \quad \text{or} \quad x + 6y + 18 = 0.$$

89. Given the vertices of a triangle: $A(0; 1)$, $B(6; 5)$ and $C(12; -1)$. Set up the equation of the altitude of the triangle drawn from the vertex C.

Solution. We make use of formula (4) of 1.2.5 to find the slope of the side AB:

$$k = \frac{5-1}{6-0} = \frac{4}{6} = \frac{2}{3}.$$

By virtue of the condition of perpendicularity, the slope of the altitude drawn from the vertex C is equal to $-3/2$. The equation of that altitude is of the form

$$y + 1 = -\frac{3}{2}(x - 12), \quad \text{or} \quad 3x + 2y - 34 = 0.$$

90. Given the sides of a triangle: $x + 3y - 7 = 0\,(AB)$, $4x - y - 2 = 0\,(BC)$, $6x + 8y - 35 = 0\,(AC)$. Find the length of the altitude drawn from the vertex B.

Solution. Let us determine the coordinates of the point B. Solving the system of equations $x + 3y - 7 = 0$ and $4x - y - 2 = 0$, we get $x = 1$, $y = 2$, i.e. $B(1; 2)$. We find the length of the altitude BB_1 as the distance from the point B to the line AC:

$$|BB_1| = \frac{|6 \cdot 1 + 8 \cdot 2 - 35|}{\sqrt{6^2 + 8^2}} = 1.3.$$

91. Determine the distance between the parallel straight lines $3x + y - 3\sqrt{10} = 0$ and $6x + 2x + 5\sqrt{10} = 0$.

Solution. The problem reduces to finding the distance from an arbitrary point of one line to the other line. Putting $x = 0$, for instance, in the equation of the first line, we obtain $y = 3\sqrt{10}$. Thus $M(0; 3\sqrt{10})$ is a point lying on the first line. Determining now the distance from the point M to the second line, we get

$$d = \frac{|6 \cdot 0 + 2 \cdot 3\sqrt{10} + 5\sqrt{10}|}{\sqrt{36 + 4}} = \frac{11\sqrt{10}}{2\sqrt{10}} = 5.5.$$

92. Set up the equations of the bisectors of the angles between the lines $x + y - 5 = 0$ and $7x - y - 19 = 0$ (Fig. 8).

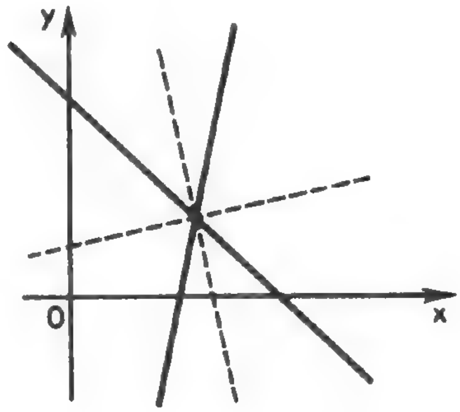

Fig. 8

Solution. Let us first solve the problem in the general form. As is known, the bisectors of the angles formed by two lines are the sets of points equidistant from these lines. If the equations of the given lines are $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ ($A_1/A_2 \neq B_1/B_2$, i.e. the lines are nonparallel), then for every point $M(\bar{x}; \bar{y})$ lying on one of the bisectors we have (using the formula for determining a distance from a point to a line)

$$\frac{|A_1\bar{x} + B_1\bar{y} + C_1|}{\sqrt{A_1^2 + B_1^2}} = \frac{|A_2\bar{x} + B_2\bar{y} + C_2|}{\sqrt{A_2^2 + B_2^2}}.$$

Since $M(\bar{x}; \bar{y})$ is an arbitrary point of the bisector, it can be simply designated as $M(x; y)$. Taking into account that the absolute values of the expressions in the last equality may be of unlike signs, we obtain for one of the bisectors the equation

$$\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}} = \frac{A_2x + B_2y + C_2}{\sqrt{A_2^2 + B_2^2}},$$

and for the other, the equation

$$\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}} = -\frac{A_2x + B_2y + C_2}{\sqrt{A_2^2 + B_2^2}}.$$

Thus, the equations of the two bisectors can be written in the form

$$\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}} \pm \frac{A_2x + B_2y + C_2}{\sqrt{A_2^2 + B_2^2}} = 0.$$

Let us now solve the specific problem on hand. Replacing $A_1, B_1, C_1, A_2, B_2, C_2$ by their values from the equations of the given lines, we get

$$\frac{x + y - 5}{\sqrt{1 + 1}} \pm \frac{7x - y - 19}{\sqrt{49 + 1}} = 0, \quad \text{i.e.} \quad 5(x + y - 5) \pm (7x - y - 19) = 0.$$

The equation of one of the bisectors is written in the form

$$5(x + y - 5) + (7x - y - 19) = 0, \quad \text{i.e.} \quad 3x + y - 11 = 0,$$

and the equation of the other, in the form

$$5(x + y - 5) - (7x - y - 19) = 0, \quad \text{i.e.} \quad x - 3y + 3 = 0.$$

93. Given the vertices of a triangle: $A(1; 1)$, $B(10; 13)$, $C(13; 6)$. Set up the equation of the bisector of the angle A.

Solution. We shall now employ another (compared with the solution of the previous problem) method of setting up an equation of the bisector.

Assume that D is a point of intersection of the bisector and the side BC. From the property of the bisector of an interior angle of a triangle it follows that $|BD| : |DC| = |AB| : |AC|$. But

$$|AB| = \sqrt{(10 - 1)^2 + (13 - 1)^2} = 15, \quad |AC| = \sqrt{(13 - 1)^2 + (6 - 1)^2} = 13.$$

Consequently, $\lambda = |BD| : |DC| = 15/13$. Since the ratio in which the point D divides the line segment BC is known, the coordinates of the point D are determined by the equalities

$$x = \frac{10 + 15/13 \cdot 13}{1 + 15/13}, \quad y = \frac{13 + 15/13 \cdot 6}{1 + 15/13},$$

or $x = 325/28$, $y = 259/28$, i.e. $D(325/28; 259/28)$. The problem reduces to deriving the equation of the line passing through the points A and D:

$$\frac{y - 1}{259/28 - 1} = \frac{x - 1}{325/28 - 1}, \quad \text{i.e.} \quad 7x - 9y + 2 = 0.$$

94. Given the equations of the altitudes of the triangle ABC: $x + y - 2 = 0$, $9x - 3y - 4 = 0$ and the coordinates of the vertex $A(2; 2)$. Derive the equations of the sides of the triangle.

Solution. It is easy to verify that the vertex A does not lie on any of the given altitudes: its coordinates do not satisfy the equations of these altitudes.

Suppose $9x - 3y - 4 = 0$ is the equation of the altitude BB_1 and $x + y - 2 = 0$ is the equation of the altitude CC_1. Let us derive the equation of the side AC considering it to be the line which passes through the point A and is

perpendicular to the altitude BB_1. Since the slope of the altitude BB_1 is 3, the slope of the side AC is $-1/3$, i.e. $k_{AC} = -1/3$. Making use of the equation of a line passing through a given point and having a given slope, we obtain the equation of the side AC:

$$y - 2 = -\frac{1}{3}(x - 2), \quad \text{or} \quad x + 3y - 8 = 0.$$

By analogy we get $k_{CC_1} = -1$, $k_{AB} = 1$, and the equation of the side AB is of the form

$$y - 2 = x - 2, \quad \text{i.e.} \quad y = x.$$

Solving simultaneously the equations of the lines AB and BB_1 and of the lines AC and CC_1, we find the coordinates of the vertices of the triangle: $B(2/3;\ 2/3)$ and $C(-1;\ 3)$. It remains to derive the equation of the side BC:

$$\frac{y - 2/3}{3 - 2/3} = \frac{x - 2/3}{-1 - 2/3}, \quad \text{i.e.} \quad 7x + 5y - 8 = 0.$$

95. Set up the equations of the lines passing through the point $M(5;\ 1)$ and forming an angle $\pi/4$ with the line $2x + y - 4 = 0$ (Fig. 9).

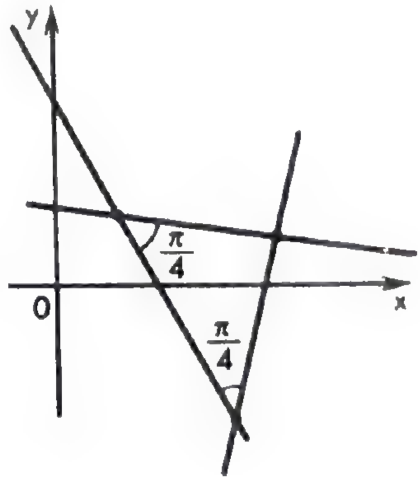

Fig. 9

Solution. Assume that the slope of one of the desired lines is k. The slope of the given line is equal to -2. Since the angle between these lines is equal to $\pi/4$, we get

$$\tan\frac{\pi}{4} = \left|\frac{k + 2}{1 - 2k}\right|, \quad \text{i.e.} \quad 1 = \left|\frac{k + 2}{1 - 2k}\right|,$$

whence

$$\frac{k + 2}{1 - 2k} = 1 \quad \text{and} \quad \frac{k + 2}{1 - 2k} = -1.$$

Solving each equation obtained, we find that $k = -1/3$ and $k = 3$. Thus the equation of one of the desired lines can be written in the form

$$y - 1 = -\frac{1}{3}(x - 5), \quad \text{i.e.} \quad x + 3y - 8 = 0,$$

and the equation of the other line, in the form

$$y - 1 = 3(x - 5), \quad \text{i.e.} \quad 3x - y - 14 = 0.$$

96. Find the line belonging to the pencil $2x + 3y + 5 + \lambda(x + 8y + 6) = 0$ and passing through the point $M(1; 1)$.

Solution. The coordinates of the point M must satisfy the equation of the desired line; therefore, to determine λ we obtain the equation

$$2 \cdot 1 + 3 \cdot 1 + 5 + \lambda(1 + 8 \cdot 1 + 6) = 0, \text{ or } 10 + 15\lambda = 0, \text{ i.e. } \lambda = -2/3.$$

Substituting the value of λ in the equation of the pencil of lines, we obtain the equation of the desired line:

$$2x + 3y + 5 - \frac{2}{3}(x + 8y + 6) = 0, \quad \text{or} \quad 4x - 7y + 3 = 0.$$

97. Find the straight line which passes through the point of intersection of the lines $3x - 4y + 7 = 0$ and $5x + 2y + 3 = 0$ and is parallel to the axis of ordinates.

Solution. The line belongs to the pencil

$$3x - 4y + 7 + \lambda(5x + 2y + 3) = 0, \text{ i.e } (3 + 5\lambda)x + (-4 + 2\lambda)y + (7 + 3\lambda) = 0.$$

Since the sought-for line is parallel to the axis of ordinates, the coefficient in y must be equal to zero: $-4 + 2\lambda = 0$, i.e. $\lambda = 2$.

It remains to substitute the value of λ we have found in the equation of the pencil to obtain the desired equation $x + 1 = 0$.

98. Given the sides of a triangle: $x + 2y + 5 = 0\,(AB)$, $3x + y + 1 = 0\,(BC)$ and $x + y + 7 = 0\,(AC)$. Set up the equation of the altitude of the triangle dropped to the side AC.

Solution. The altitude belongs to the pencil

$$x + 2y + 5 + \lambda(3x + y + 1) = 0, \quad \text{i.e.} \quad (1 + 3\lambda)x + (2 + \lambda)y + (5 + \lambda) = 0.$$

The slope of the line belonging to the pencil is equal to $-(1 + 3\lambda)/(2 + \lambda)$. Since the slope of the line AC is equal to -1, the slope of the desired altitude is unity, and we get for λ the equation $-(1 + 3\lambda)/(2 + \lambda) = 1$. Hence $1 + 3\lambda + 2 + \lambda = 0$, i.e. $\lambda = -3/4$. Substituting the value of λ we have obtained into the equation of the pencil of lines, we get the desired equation of the altitude:

$$\left(1-\frac{9}{4}\right)x+\left(2-\frac{3}{4}\right)y+\left(5-\frac{3}{4}\right)=0, \text{ i.e. } 5x-5y-17=0.$$

99. Given the vertices of the triangle ABC: $A(0; 2)$, $B(7; 3)$ and $C(1; 6)$. Determine $\widehat{BAC}=\alpha$.

100. Given the sides of a triangle: $x+y-6=0$, $3x-5y+14=0$ and $5x-3y-14=0$. Set up equations of its altitudes.

101. Set up the equations of the bisectors of the angles between the straight lines $3x+4y-20=0$ and $8x+6y-5=0$.

102. Given the vertices of a triangle: $A(0; 0)$, $B(-1; -3)$ and $C(-5; -1)$. Set up the equations of the straight lines which pass through the vertices of the triangle and are parallel to its sides.

103. Set up the equations of the straight lines passing through the point $M(2; 7)$ and making angles of 45° with the line AB, where $A(-1; 7)$ and $B(8; -2)$.

104. Determine the distance from the point $M(2; -1)$ to the straight line intercepting the line segments $a=8$, $b=6$ on the coordinate axes.

105. In a triangle with the vertices $A(3/2; 1)$, $B(1; 5/3)$, $C(3; 3)$ find length of the altitude drawn from the vertex C.

106. At what value of m do the lines $7x-2y-5=0$, $x+7y-8=0$ and $mx+my-8=0$ intersect at one point?

107. Given the midpoints of the sides of a triangle $A_1(-1; -1)$, $B_1(1; 9)$ and $C(9; 1)$. Set up the equations of the midpoint perpendiculars to the sides of the triangle.

108. Find the acute angle formed by the ordinate axis and the line passing through the points $A(2; \sqrt{3})$ and $B(3; 2\sqrt{3})$.

109. The points $A(1; 2)$ and $C(3; 6)$ are the opposite vertices of a square. Determine the coordinates of the other two vertices of the square.

110. Find the point on the abscissa axis whose distance from the line $8x+15y+10=0$ is unity.

111. Given the vertices of a triangle: $A(1; 1)$, $B(4; 5)$ and $C(13; -4)$. Set up the equations of the median drawn from the vertex B and of the altitude dropped from the vertex C. Compute the area of the triangle.

112. Find the straight lines which belong to the pencil $2x+3y+6+\lambda(x-5y-6)=0$ and are at right angles to the basic lines of the pencil.

113. Find the line passing through the point of intersection of the lines $x+6y+5=0$, $3x-2y+1=0$ and through the point $M(-4/5; 1)$.

114. Find the line which passes through the point of intersection of the lines $x+2y+3=0$, $2x+3y+4=0$ and is parallel to the line $5x+8y=0$.

115. Find the line which passes through the point of intersection of the lines $3x-y-1=0$, $x+3y+1=0$ and is parallel to the abscissa axis.

116. Find the line passing through the point of intersection of the lines $5x+3y+10=0$, $x+y-15=0$ and through the origin.

117. Find the line passing through the point of intersection of the lines $x+2y+1=0$, $2x+y+2=0$ and making an angle of 135° with the abscissa axis.

118. Set up the equations of the straight lines passing through the point $M(a; b)$ and making an angle of 45° with the line $x + y + c = 0$.

119. Given the sides of a triangle: $x - y = 0\,(AB), x + y - 2 = 0\,(BC), y = 0\,(AC)$. Set up the equations of the median passing through the vertex B and of the altitude passing through the vertex A.

120. Show that the triangle with the sides $x + y\sqrt{3} + 1 = 0, x\sqrt{3} + y + 1 = 0$ and $x - y - 10 = 0$ is an isosceles one. Find its vertex angle.

121. Given the consecutive vertices of a parallelogram: $A(0; 0)$, $B(1; 3)$, $C(7; 1)$. Find the angle between its diagonals and show that the parallelogram is a rectangle.

122. Given the sides of a triangle: $x - y + 2 = 0\,(AB)$, $x = 2\,(BC)$, $x + y - 2 = 0\,(AC)$. Set up the equation of the straight line passing through the vertex B and through the point on the side AC dividing it (reckoning from the vertex A) in the ratio $1:3$.

123. Show that the triangle with the vertices $A(1; 1)$, $B(2; 1 + \sqrt{3})$, $C(3; 1)$ is equilateral and compute its area.

124. Show that the triangle whose sides are specified by equations with integral coefficients cannot be equilateral.

125. Given the vertex $A(3; 9)$ of a triangle and the equations of the medians: $y - 6 = 0$ and $3x - 4y + 9 = 0$. Find the coordinates of the other two vertices.

126. Set up the equation of the hypotenuse of a right-angled triangle passing through the point $M(2; 3)$ if the sides of the triangle lie on the coordinate axes and the area of the triangle is equal to 12 square units.

127. Set up the equations of three sides of a square if the fourth side is known to be a segment of the line $4x + 3y - 12 = 0$ whose end points lie on the coordinate axes.

1.3. Quadratic Curves

1.3.1. A circle. A *circle* is a set of points equidistant from a given point (a centre). If r is the radius of the circle and the point $C(a; b)$ is its centre, then the equation of the circle is of the form

$$(x - a)^2 + (y - b)^2 = r^2. \tag{1}$$

In particular, if the centre of the circle coincides with the origin, then the indicated equation has the form

$$x^2 + y^2 = r^2.$$

If we remove the parentheses on the left-hand side of equation (1), we obtain an equation of the form

$$x^2 + y^2 + lx + my + n = 0, \tag{2}$$

where $l = -2a, m = -2b, n = a^2 + b^2 - r^2$.

In a general case equation (2) specifies a circle if $l^2 + m^2 - 4n > 0$.

If $l^2 + m^2 - 4n = 0$, then the indicated equation specifies the point $(-l/2; -m/2)$, and if $l^2 + m^2 - 4n < 0$, it is meaningless from the point of view of geometry. In that case it is said to specify an imaginary circle.

It may be of use to recall that the equation of a circle contains the leading terms x^2 and y^2 with equal coefficients and lacks the term with the product of x by y.

The mutual positions of the point $M(x_1; y_1)$ and the circle $x^2 + y^2 = r^2$ depends on the following conditions: if $x_1^2 + y_1^2 = r^2$, then the point M lies on the circle; if $x_1^2 + y_1^2 > r^2$, then the point M is outside the circle, now if $x_1^2 + y_1^2 < r^2$, then the point M is inside the circle.

128. Find the coordinates of the centre and the radius of the circle $2x^2 + 2y^2 - 8x + 5y - 4 = 0$.

Solution. Dividing the equation by 2 and collecting like terms, we obtain $x^2 - 4x + y^2 + \frac{5}{2}y = 2$. Let us complete the expressions $x^2 - 4x$ and $y^2 + \frac{5}{2}y$ to perfect squares by adding 4 to the first binomial and $(5/4)^2$ to the second binomial (at the same time the sum of these numbers is added to the right-hand side):

$$(x^2 - 4x + 4) + \left(y^2 + \frac{5}{2}y + \frac{25}{16}\right) = 2 + 4 + \frac{25}{16} \text{ or } (x-2)^2 + \left(y + \frac{5}{4}\right)^2 = \frac{121}{16}.$$

Thus the coordinates of the centre of the circle are $a = 2$, $b = -5/4$ and the radius of the circle is $r = 11/4$.

129. Set up the equation of the circle circumscribed about a triangle whose sides are specified by the equations $9x - 2y - 41 = 0$, $7x + 4y + 7 = 0$, $x - 3y + 1 = 0$.

Solution. Let s find the coordinates of the vertices of a triangle by solving simultaneously three systems of equations:

$$\begin{cases} 9x - 2y - 41 = 0, \\ 7x + 4y + 7 = 0; \end{cases} \quad \begin{cases} 9x - 2y - 41 = 0, \\ x - 3y + 1 = 0; \end{cases} \quad \begin{cases} 7x + 4y + 7 = 0, \\ x - 3y + 1 = 0. \end{cases}$$

The result will be $A(3; -7)$, $B(5; 2)$, $C(-1; 0)$.

Suppose the desired equation has the form $(x - a)^2 + (y - b)^2 = r^2$. To find a, b and r, let us write three equalities substituting in the sought-for equation the coordinates of the points A, B and C for the current coordinates:

$$(3 - a)^2 + (-7 - b)^2 = r^2; \ (5 - a)^2 + (2 - b)^2 = r^2;$$
$$(-1 - a)^2 + b^2 = r^2.$$

Eliminating r^2, we arrive at the following system of equations:

$$\begin{cases} (3 - a)^2 + (-7 - b)^2 = (5 - a)^2 + (2 - b)^2, \\ (3 - a)^2 + (-7 - b)^2 = (-1 - a)^2 + b^2, \end{cases} \text{ or } \begin{cases} 4a + 18b = -29, \\ 8a - 14b = 57. \end{cases}$$

Hence $a = 3.1$, $b = -2.3$. The value of r^2 can be found from the equation $(-1 - a)^2 + b^2 = r^2$, i.e. $r^2 = 22.1$. Thus the desired equation can be written in

the form

$$(x - 3.1)^2 + (y + 2.3)^2 = 22.1.$$

130. Set up the equation of the circle passing through the points $A(5; 0)$ and $B(1; 4)$ if its centre lies on the line $x + y - 3 = 0$.

Solution. Let us find the coordinates of the point M which is the midpoint of the chord AB; we have $x_M = (5 + 1)/2 = 3$, $y_M = (4 + 0)/2 = 2$, i.e. $M(3; 2)$. The centre of the circle lies on the perpendicular to the midpoint of the closed interval $[AB]$. The equation of the line (AB) has the form

$$(y - 0)/(4 - 0) = (x - 5)/(1 - 5), \quad \text{i.e.} \quad x + y - 5 = 0.$$

Since the slope of this line is -1, the slope of the perpendicular to that line is equal to 1, and the equation of the perpendicular is

$$y - 2 = 1 \cdot (x - 3), \quad \text{i.e.} \quad x - y - 1 = 0.$$

The centre of the circle C is, evidently, the point of intersection of the line (AB) with the indicated perpendicular, that is, the coordinates of the centre can be determined by solving the system of equations $x + y - 5 = 0$, $x - y - 1 = 0$. Consequently, $x = 2$, $y = 1$, i.e. $C(2; 1)$. The radius of the circle is equal to the length of the closed interval $[CA]$, i.e. $r = \sqrt{(5 - 2)^2 + (1 - 0)^2} = \sqrt{10}$. Thus, the sought-for equation has the form

$$(x - 2)^2 + (y - 1)^2 = 10.$$

131. Set up the equation of the chord of the circle $x^2 + y^2 = 49$ which is divided in two at the point $A(1; 2)$.

Solution. Let us set up the equation of the diameter of the circle passing through the point $A(1; 2)$. This equation has the form $y = 2x$. The sought-for chord is perpendicular to the diameter and passes through the point A, that is its equation is

$$y - 2 = -\frac{1}{2}(x - 1), \quad \text{or} \quad x + 2y - 5 = 0.$$

132. Find the equation of the circle symmetric with respect to the circle $x^2 + y^2 = 2x + 4y - 4$ about the line $x - y - 3 = 0$.

Solution. Reduce the equation of the given circle to the canonical form $(x - 1)^2 + (y - 2)^2 = 1$; the centre of the circle is at the point $C(1; 2)$ and its radius is equal to 1. Find the coordinates of the centre $C_1(x_1; y_1)$ of the symmetric circle, drawing for that purpose a line through the point $C(1; 2)$ at right angles to the line $x - y - 3 = 0$. Its equation is $y - 2 = k(x - 1)$, where $k = -1/1 = -1$, whence

$$y - 2 = -x + 1, \quad \text{or} \quad x + y - 3 = 0.$$

Simultaneous solution of the equations $x - y - 3 = 0$ and $x + y - 3 = 0$ yields $x = 3$, $y = 0$, that is, the projection of the point $C(1; 2)$ on the given straight line is the point $P(3; 0)$. As to the coordinates of the symmetric point, they can be obtained by the formulas specifying the coordinates of the midpoint of the segment:

$3 = (1 + x_1)/2$, $0 = (2 + y_1)/2$; thus, $x_1 = 5$, $y_1 = -2$. This means that the point $C_1(5; -2)$ is the centre of the symmetric circle and the equation of that circle has the form

$$(x - 5)^2 + (y + 2)^2 = 1.$$

133. Find the set of the midpoints of the chords of the circle $x^2 + y^2 = 4(y + 1)$ drawn through the origin.

Solution. The equation of a set of chords has the form $y = kx$. Let us express the coordinates of the point of intersection of the chords with the circle as k, for which purpose we shall solve the system of equations $y = kx$ and $x^2 + y^2 - 4y - 4 = 0$ and obtain a quadratic equation $x^2(k^2 + 1) - 4kx - 4 = 0$. Here $x_1 + x_2 = 4k/(1 + k^2)$. But the half-sum of these abscissas yields the abscissa of the midpoint of the chord, i.e. $x = 2k/(1 + k^2)$, and the ordinate of the midpoint of the chord is $y = 2k^2/(1 + k^2)$. The last two equalities are the parametric equations of the desired set of points.

Eliminating k from these equations (for which purpose it is sufficient to put $k = y/x$ in the relation $x = 2k(1 + k^2)$, we get $x^2 + y^2 - 2y = 0$. Thus the sought-for set is the circle as well.

134. Determine the coordinates of the centres and the radii of the circles: (1) $x^2 + y^2 - 8x + 6y = 0$, (2) $x^2 + y^2 + 10x - 4y + 29 = 0$, (3) $x^2 + y^2 - 4x + 14y + 54 = 0$.

135. Find the angle between the radii of the circle $x^2 + y^2 + 4x - 6y = 0$ drawn to the points of its intersection with the Oy axis.

136. Set up an equation of the circle passing through the points $A(1; 2)$, $B(0; -1)$ and $C(-3; 0)$.

137. Set up the equation of the circle passing through the points $A(7; 7)$, $B(-2; 4)$ if its centre lies on the line $2x - y - 2 = 0$.

138. Set up the equation of the common chord of the circles $x^2 + y^2 = 16$ and $(x - 5)^2 + y^2 = 9$.

139. Set up the equations of the tangent lines to the circle $(x - 3)^2 + (y + 2)^2 = 25$ drawn at the points of intersection of the circle and the straight line $x - y + 2 = 0$.

140. Given the circle $x^2 + y^2 = 4$. A chord AB is drawn from the point $A(-2; 0)$ and is extended to the distance $|BM| = |AB|$. Find the set of points M.

1.3.2. An ellipse. An *ellipse* is a set of points the sum of whose distances from the two given points, called the *foci*, is constant (it is designated by $2a$), this constant being larger than the distance between the foci.

If the position of the coordinate axes with respect to an ellipse is such as is shown in Fig. 10 and the foci of the ellipse are on the x-axis at the points $F_1(c; 0)$ and $F_2(-c; 0)$ at equal distances from the origin, then we have the *simplest (canonical) equation of an ellipse:*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1}$$

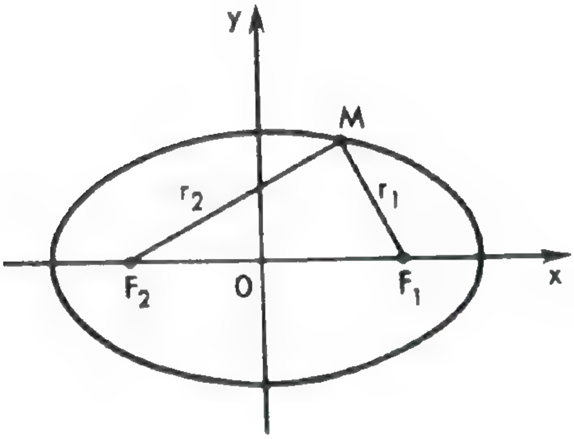

Fig. 10

Here a is the semi-major and b, the semi-minor axis of the ellipse, a, b and c(c is half the distance between the foci) being connected by the relation $a^2 = b^2 + c^2$.

The shape of an ellipse (the degree of its contraction) is characterized by its *eccentricity* $e = c/a$ (since $c < a$, we have $e < 1$).

The distances of some point M of an ellipse from its foci are called the *focal radius-vectors* of that point. It is customary to designate them as r_1 and r_2 (by virtue of the definition of an ellipse, $r_1 + r_2 = 2a$ for its any point).

In a special case, when $a = b$($c = 0$, $e = 0$, the foci merge at a single point, the centre) an ellipse becomes a circle (with the equation $x^2 + y^2 = a^2$).

The mutual position of the point $M(x_1; y_1)$ and the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is determined by the following conditions: if $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$, then the point M lies on the ellipse; if $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} > 1$, then the point M is outside the ellipse, now if $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} < 1$, then the point M is inside the ellipse.

The focal radius-vectors are expressed in terms of the abscissa of the point of the ellipse by the formulas $r_1 = a - ex$ (the right-hand focal radius-vector) and $r_2 = a + ex$ (the left-hand focal radius-vector).

141. Set up the canonical equation of the ellipse passing through the points $M(5/2; \sqrt{6}/4)$ and $N(-2; \sqrt{15}/5)$.

Solution. Suppose $x^2/a^2 + y^2/b^2 = 1$ is the desired equation of the ellipse. This equation must be satisfied by the coordinates of the given points. Consequently

$$\frac{25}{4a^2} + \frac{3}{8b^2} = 1, \quad \frac{4}{a^2} + \frac{3}{5b^2} = 1,$$

whence we find $a^2 = 10$, $b^2 = 1$. Thus, the equation of the ellipse has the form

$$\frac{x^2}{10} + y^2 = 1.$$

142. Find the point on the ellipse $x^2/25 + y^2/9 = 1$ the difference between whose focal radius-vectors is equal to 6.4.

143. Find the length of the perpendicular erected from the focus of the ellipse $x^2/a^2 + y^2/b^2 = 1$ to the major axis till the intersection with the ellipse.

144. Set up the equation of the line passing through the left-hand focus and the lower vertex of the ellipse $x^2/25 + y^2/16 = 1$.

145. The ellipse considered relative to the axes passes through the point $M(1; 1)$ and has the eccentricity $e = 3/5$. Set up the equation of the ellipse.

146. What are the positions of the points $M(7; 1)$, $N(-5; -4)$, $P(4; 5)$ with respect to the ellipse $x^2/50 + y^2/32 = 1$?

147. Find the eccentricity of the ellipse if the focal segment can be seen from the upper vertex at the angle α.

148. Find the point on the line $x + 5 = 0$ equidistant from the left focus and the upper vertex of the ellipse $x^2/20 + y^2/4 = 1$.

149. Using the definition of an ellipse, set up its equation if it is known that the points $F_1(0; 0)$ and $F_2(1; 1)$ are the foci of the ellipse and the length of the major axis is equal to 2.

150. Set up the equation of the set of points whose distances from the point $A(0; 1)$ constitute half the distance to the line $y - 4 = 0$.

151. The end points of the segment AB of constant length a are slipping along the sides of a right angle. Find the equation of the curve described by the point M which divides the segment in the ratio 1 : 2.

1.3.3. A hyperbola. A *hyperbola* is a set of points the absolute value of the difference of whose distances from two given points, called *foci*, is a constant quantity (it is usually denoted by $2a$), the constant quantity being smaller than the distance between the foci. If we place the foci of a hyperbola at the points $F_1(c; 0)$ and $F_2(-c; 0)$, we obtain the *canonical equation of the hyperbola*:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{1}$$

where $b^2 = c^2 - a^2$. A hyperbola consists of two branches and is symmetric about the coordinate axes. The points $A_1(a; 0)$ and $A_2(-a; 0)$ are called the *vertices* of a hyperbola. The line segment $|A_1A_2| = 2a$ is known as the *transverse (real) axis* of a hyperbola and the segment $|B_1B_2| = 2b$, as the *conjugate axis* (Fig. 11).

A line is an *asymptote* to a hyperbola if the distance of the point $M(x; y)$ of the hyperbola to that line tends to zero as $x \to +\infty$ or $x \to -\infty$. A hyperbola possesses two asymptotes whose equations are $y = \pm(b/a)x$.

To construct the asymptotes to a hyperbola, it is necessary to construct a rectangle on the axes of the hyperbola with the sides $x = a, x = -a, y = b, y = -b$. The lines passing through the opposite vertices of that rectangle are the asymptotes to the hyperbola. Figure 11 shows the mutual position of the hyperbola and its asymptotes. The ratio $e = c/a > 1$ is called the *eccentricity* of a hyperbola.

The focal radius-vectors of the right-hand branch of the hyperbola are $r_1 = ex - a$ (the right focal radius-vector), $r_2 = ex + a$ (the left focal radius-vector).

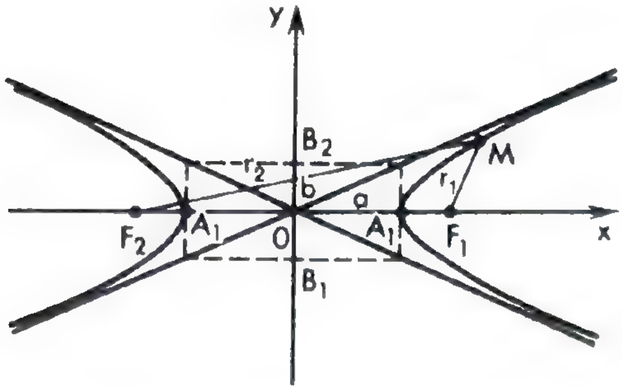

Fig. 11

The focal radius-vectors of the left-hand branch of the hyperbola are $r_1 = -ex + a$ (the right focal radius-vector), $r_2 = -ex - a$ (the left focal radius-vector).

If $a = b$, the equation of the hyperbola assumes the form

$$x^2 - y^2 = a^2.$$

Such a hyperbola is called *equilateral*. Its asymptotes form a right angle. If the asymptotes to an equilateral hyperbola are taken to be the coordinate axes, then its equation assumes the form $xy = m$ ($m = \pm a^2/2$; at $m > 0$ the hyperbola is located in the first and third quadrants, at $m < 0$, in the second and fourth quadrants). Since the equation $xy = m$ can be rewritten in the form $y = m/x$, an equilateral hyperbola is the graph of the inversely proportional relationship between the quantities x and y.

The equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \left(\text{or } \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1\right) \qquad (2)$$

is also an equation of a hyperbola, but the transverse axis of this hyperbola is the intercept of the Oy axis $2b$ in length.

Two hyperbolas $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$ possess the same semi-axes and the same asymptotes, but the transverse axis of one of them serves as the conjugate axis of the other and vice versa. Two hyperbolas of this kind are called *conjugate* hyperbolas.

152. Find the point on the right-hand branch of the hyperbola $x^2/16 - y^2/9 = 1$ whose distance from the right focus is half that from the left focus.

Solution. For the right-hand branch of a hyperbola, the focal radius-vectors can be determined by the formulas $r_1 = ex - a$ and $r_2 = ex + a$. Consequently, we have an equation $ex + a = 2(ex - a)$, whence $x = 3a/e$; here $a = 4$, $e = c/a = \sqrt{a^2 + b^2}/a = \sqrt{16 + 9}/4 = 5/4$, i.e. $x = 9.6$.

The ordinate can be found from the equation of the hyperbola

$$y = \pm\frac{3}{4}\sqrt{x^2 - 16} = \pm\frac{3}{4}\sqrt{\left(\frac{48}{5}\right)^2 - 16} = \pm\frac{3}{5}\sqrt{119}.$$

Thus, two points satisfy the conditions of the problem: $M_1(9.6;\ 0.6\sqrt{119})$ and $M_2(9.6;\ -0.6\sqrt{119})$.

153. Given the points $A(-1;\ 0)$ and $B(2;\ 0)$. A point M is moving in such a way that the angle $\hat{B}$ in the triangle AMB remains twice as large as the angle $\hat{A}$. Find the equation of the curve described by the point M.

Solution. Taking the point M with the coordinates x and y, we express $\tan\hat{B}$ and $\tan\hat{A}$ in terms of the coordinates of the points A, B and M:

$$\tan\hat{B} = -\frac{y}{x-2} = \frac{y}{2-x}, \quad \tan\hat{A} = \frac{y}{x+1}.$$

By the condition of the problem, we get an equation $\tan\hat{B} = \tan 2\hat{A}$, i.e. $\tan\hat{B} = 2\tan\hat{A}/(1 - \tan^2\hat{A})$. Substituting the expressions obtained for $\tan\hat{B}$ and $\tan\hat{A}$ into this equality, we arrive at the equation

$$\frac{y}{2-x} = \frac{2y/(x+1)}{1 - y^2/(1+x)^2};$$

after cancelling out $y\ (y \neq 0)$ and simplifying, we obtain $x^2 - y^2/3 = 1$. The sought-for curve is a hyperbola.

154. The eccentricity of a hyperbola is equal to $\sqrt{2}$. Set up the simplest equation of a hyperbola passing through the point $M(\sqrt{3};\ \sqrt{2})$.

Solution. In accordance with the definition of eccentricity, we have $c/a = \sqrt{2}$, or $c^2 = 2a^2$. But $c^2 = a^2 + b^2$. Consequently, $a^2 + b^2 = 2a^2$, or $a^2 = b^2$, that is, the hyperbola is equilateral.

Another equality can be obtained from the condition for finding the point M on the hyperbola, i.e. $(\sqrt{3})^2/a^2 - (\sqrt{2})^2/b^2 = 1$, or $3/a^2 - 2/b^2 = 1$. Since $a^2 = b^2$, we have $3/a^2 - 2/a^2 = 1$, i.e. $a^2 = 1$.

Thus, the equation of the sought-for parabola is of the form $x^2 - y^2 = 1$.

155. Set up the equation of the hyperbola passing through the point $M(9;\ 8)$ if the asymptotes to the hyperbola satisfy the equations $y = \pm(2\sqrt{2}/3)x$.

156. Find the equation of the hyperbola whose vertices and foci coincide with the respective foci and vertices of the ellipse $x^2/8 + y^2/5 = 1$.

157. A straight line is drawn through the point $M(0;\ -1)$ and the right-hand vertex of the hyperbola $3x^2 - 4y^2 = 12$. Find the second point of intersection of the line and the hyperbola.

158. Given the hyperbola $x^2 - y^2 = 8$. Find the confocal ellipse passing through the point $M(4;\ 6)$.

159. Given the ellipse $9x^2 + 25y^2 = 1$. Write the equation of the confocal equilateral hyperbola.

160. The angle between the asymptotes to a hyperbola is $60°$. Compute the eccentricity of the hyperbola.

161. On the left-hand branch of the hyperbola $x^2/64 - y^2/36 = 1$ find the point whose right focal radius-vector is equal to 18.

162. Set up the equation of the hyperbola whose eccentricity is equal to 2 and whose foci coincide with the foci of the ellipse $x^2/25 + y^2/9 = 1$.

163. Find the focal radius-vectors of the hyperbola $x^2/16 - y^2/9 = 1$ at its points of intersection with the circle $x^2 + y^2 = 91$.

164. Prove that the length of a perpendicular let fall from the focus to one of the asymptotes to the hyperbola is equal to the semi-conjugate axis.

165. Prove that the product of the distances from any point of the hyperbola $x^2 - y^2 = 1$ to its asymptotes is a constant quantity.

166. Find the equation of the set of points equidistant from the circle $x^2 + 4x + y^2 = 0$ and from the point $M(2; 0)$.

1.3.4. A parabola. A *parabola* is a set of points equidistant from a given point called a *focus* and from a given line called a *directrix*. If the directrix of a parabola is a straight line $x = -p/2$ and the focus, a point $F(p/2; 0)$, then the equation of the parabola has the form

$$y^2 = 2px. \tag{1}$$

This parabola is located symmetrically about the abscissa axis (Fig. 12, where $p > 0$).

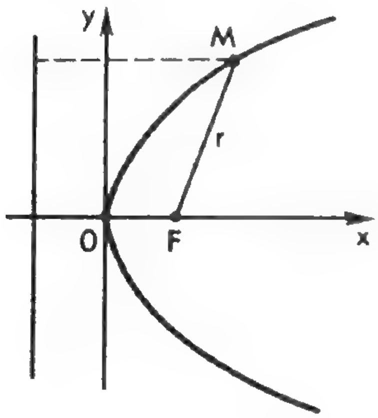

Fig. 12

The equation

$$x^2 = 2py \tag{2}$$

is the equation of a parabola which is symmetric about the axis of ordinates. At $p > 0$, parabolas (1) and (2) face the positive side of the corresponding axis and at $p < 0$, the negative side.

The length of the focal radius-vector of the parabola $y^2 = 2px$ is determined by the formula $r = x + p/2$ $(p > 0)$.

167. Set up the equation of the parabola symmetric about the Ox axis, with the vertex at the origin, if the length of a certain chord of the parabola, perpendicular to the Ox axis, is equal to 16 and the distance from the chord to the vertex is equal to 6.

Solution. Since the length of the chord and its distance from the vertex are known, the coordinates of the end point of the chord, the point M lying on the parabola, are known too. The equation of the parabola has the form $y^2 = 2px$. Putting $x = 6$ and $y = 8$ in it, we find $8^2 = 2p \cdot 6$, whence $2p = 32/3$. Thus, the equation of the desired parabola is $y^2 = 32x/3$.

168. Set up the equation of the parabola with the vertex at the origin, symmetric about the Oy axis and intercepting a chord $8\sqrt{2}$ in length on the bisector of the first and third quadrants.

Solution. The sought-for equation of the parabola is $x^2 = 2py$, the equation of the bisector is $y = x$. Thus we obtain the points of intersection of the parabola and the bisector: $O(0; 0)$ and $M(2p; 2p)$. The length of the chord can be determined as the distance between two points $8\sqrt{2} = \sqrt{4p^2 + 4p^2}$, whence $2p = 8$. Consequently, the desired equation has the form $x^2 = 8y$.

169. Set up the simplest equation of a parabola if it is known that its focus is at the point of intersection of the line $4x - 3y - 4 = 0$ and the Ox axis.

170. Find the point on the parabola $y^2 = 8x$ whose distance from the directrix is equal to 4.

171. Set up the equation of the parabola with the vertex at the origin, symmetric about the Ox axis and intercepting a chord $4\sqrt{2}$ in length on the line $y = x$.

172. The parabola $y^2 = 2x$ intercepts a chord 3/4 in length on a straight line passing through the origin. Set up the equation of that line.

173. Set up the simplest equation of a parabola if the length of the chord perpendicular to the symmetry axis and dividing in two the distance between the focus and the vertex is equal to 1.

174. Find the point on the parabola $y^2 = 32x$ whose distance from the straight line $4x + 3y + 10 = 0$ is equal to 2.

175. Set up the equation of the parabola with the vertex at the origin, symmetric about the Ox axis and passing through the point $M(4; 2)$; determine the angle α between the focal radius-vector of that point and the Ox axis.

1.4. Transformation of Coordinates and Simplification of Equations of Quadratic Curves

1.4.1. Transformation of coordinates. When passing from the xOy system of coordinates to a new system $x'O_1y'$ (the direction of the coordinate axes is the same, the point $O_1(a; b)$ is accepted as the new origin; Fig. 13), the relationship between the

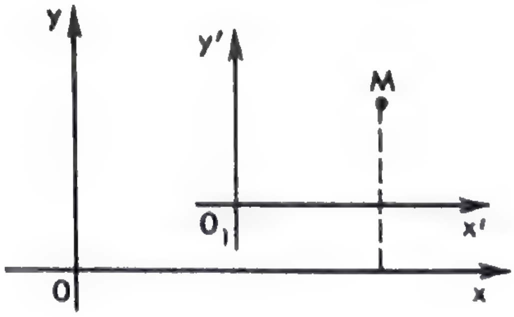

Fig. 13

old and the new coordinates of a point M of the plane is determined by the following formulas:

$$x = x' + a, \quad y = y' + b; \tag{1}$$

$$x' = x - a, \qquad y' = y - b. \tag{2}$$

By means of formulas (1) the old coordinates are expressed in terms of the new ones, and by means of formulas (2), the new coordinates are expressed in terms of the old ones.

When the coordinate axes are turned through an angle α (the origin is preserved, α being reckoned counterclockwise; Fig. 14), the relationship between the old coor-

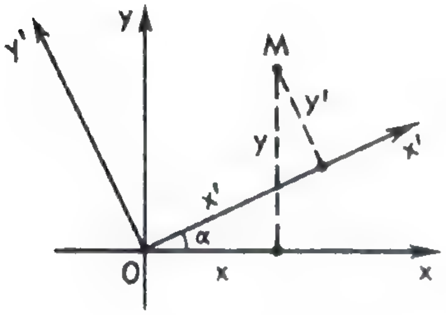

Fig. 14

dinates x, y and the new ones x', y' is specified by the following formulas:

$$x = x'\cos\alpha - y'\sin\alpha, \qquad y = x'\sin\alpha + y'\cos\alpha; \tag{3}$$

$$x' = x\cos\alpha + y\sin\alpha, \qquad y' = -x\sin\alpha + y\cos\alpha. \tag{4}$$

176. A parallel translation of the coordinate axes has been performed, the new origin being at point $O_1(3; -4)$. The old coordinates of the point $M(7; 8)$ are known. Determine the new coordinates of the point.

Solution. Here $a = 3$, $b = -4$, $x = 7$, $y = 8$. Formula (2) yields $x' = 7 - 3 = 4, y' = 8 - (-4) = 12$.

177. Given the point $M(4; 3)$ on the plane xOy. The system of coordinates has been turned about the origin so that the new axis passes through the point M. Determine the old coordinates of a point A if its new coordinates are $x' = 5, y' = 5$.

Solution. $|OM| = \sqrt{4^2 + 3^2} = 5$, we have $\sin\alpha = 3/5$, $\cos\alpha = 4/5$; then formulas (3) for the transformation of coordinates assume the following form for the given problem:

$$x = (4/5)x' - (3/5)y', \quad y = (3/5)x' + (4/5)y'.$$

Putting $x' = y' = 5$, we find $x = 1, y = 7$.

178. The coordinate system has been turned through the angle $\alpha = \pi/6$. Determine the new coordinates of the point $M(\sqrt{3}; 3)$.

Solution. Making use of formulas (4), we obtain

$$x' = \sqrt{3}\cos(\pi/6) + 3\sin(\pi/6) = 3/2 + 3/2 = 3,$$

$$y' = -\sqrt{3}\sin(\pi/6) + 3\cos(\pi/6) = -\sqrt{3}/2 + 3\sqrt{3}/2 = \sqrt{3}.$$

179. Given the point $M(9/2; 11/2)$. The straight lines $2x - 1 = 0$ (O_1y' axis) and $2y - 5 = 0$ (O_1x' axis) are taken as the new coordinate axes. Find the coordinates of the point M in the new coordinate system.

180. Given the point $M(4\sqrt{5}\,;\,2\sqrt{5})$. The line $y = 2x$ is taken as the new abscissa axis and the line $y = -0.5x$, as the new axis of ordinates, the new coordinate axes forming acute angles with the respective old axes. Find the coordinates of the point M in the new system.

1.4.2. The parabola $y = Ax^2 + Bx + C$ and the hyperbola $y' = (kx + l)/(px + q)$. The equation of the form

$$y = Ax^2 + Bx + C$$

can be reduced to the canonical equation of a parabola by means of a transformation of coordinates upon a parallel translation of the axes, that is, by the formulas $x' = x + a$, $y = y' + b$ (a and b being the coordinates of the new origin, x' and y', the new coordinates).

The parabola specified by the equation $y = Ax^2 + Bx + C$ has an axis of symmetry parallel to the Oy axis (similarly, the equation $x = Ay^2 + By + C$ specifies a parabola with the axis of symmety which is parallel to the Ox axis).

The linear-fractional function

$$y = (kx + l)/(px + q)$$

specifies an equilateral hyperbola if $kq - pl \neq 0$, $p \neq 0$; by a transformation of coordinates upon a parallel translation of the coordinate axes this equation is reduced to the canonical form of the equation of an equilateral hyperbola $xy = m$, that is, to an equation of an equilateral hyperbola whose coordinate axes are asymptotes. At $m > 0$ the branches of the hyperbola are located in the first and the third quadrant, and at $m < 0$, in the second and the fourth quadrant.

181. Reduce the equation of the parabola $y = 9x^2 - 6x + 2$ to the canonical form.

Solution. Replace x by $x' + a$ and y by $y' + b$:

$y' + b = 9(x' + a)^2 - 6(x' + a) + 2$, or $y' = 9x'^2 + 6x'(3a - 1) + (9a^2 - 6a + 2 - b)$.

Find the values of a and b at which the coefficient in x' and the constant term turn into zero: $3a - 1 = 0$, $9a^2 - 6a + 2 - b = 0$, i.e. $a = 1/3$, $b = 1$. Consequently, the canonical equation of the parabola has the form $x'^2 = (1/9)y'$. The vertex of the parabola is at the point $O_1(1/3; 1)$ and $p = 1/18$.

Another method of solving problems of this kind consists in the reduction of tne given equation of the form $y = Ax^2 + Bx + C$ (or $x = Ay^2 + By + C$) to the form $(x - a)^2 = 2p(y - b)$ [$(y - b)^2 = 2p(x - a)$ respectively]. Then the point $O_1(a; b)$ serves as the vertex of the parabola, and the sign of the parameter p determines the side, positive or negative, of the corresponding axis (Oy or Ox) to which the parabola extends.

Thus, the equation $y = 9x^2 - 6x + 2$ can be transformed in the following way:

$$y = 9\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) - 1 + 2;\quad y - 1 = 9\left(x - \frac{1}{3}\right)^2;\quad \left(x - \frac{1}{3}\right)^2 = \frac{1}{9}(y - 1).$$

It again follows from this equation that the vertex of the parabola is at the point $O_1(1/3; 1)$, the parameter $p = 1/18$, and the branch of the parabola extends to the positive side of the Oy axis.

182. Reduce the equation of the hyperbola $y = (4x + 5)/(2x - 1)$ to the form $x'y' = k$. Find the equations of the asymptotes to the hyperbola with respect to the original system of coordinates.

Solution. By means of a parallel translation of the coordinate axes we bring the given equation to the form

$$(y' + b)(2x' + 2a - 1) = 4x' + 4a + 5,$$

or

$$2x'y' + (2b - 4)x' + (2a - 1)y' = 4a + b - 2ab + 5.$$

Let us find a and b from the conditions $2b - 4 = 0$ and $2a - 1 = 0$, i.e. $a = 0.5$, $b = 2$. Then, in the new coordinate system the equation of the hyperbola assumes the form $x'y' = 3.5$. The asymptotes to the hyperbola are the new coordinate axes and, therefore, their equations are $x' = 0.5$ and $y' = 2$.

Another method of solving problems of this kind consists in the following: the equation of the form $y = (kx + l)/(px + q)$ is transformed into $(x - a)(y - b) = m$; the centre of the hyperbola is at the point $O_1(a; b)$; its asymptotes are the straight lines $x = a$ and $y = b$, the sign of m again determines the angles between the asymptotes which contain the branches of the hyperbola.

Thus, the equation $y = (4x + 5)/(2x - 1)$ can be transformed as follows:

$$2\left(x-\frac{1}{2}\right)y-4\left(x-\frac{1}{2}+\frac{7}{4}\right) = 0; (2x - 1)y - (4x + 5) = 0; 2(x - 0.5)(y - 2) = 7.$$

This means that the equation of the hyperbola has been reduced to the form $(x - 0.5)(y - 2) = 3.5$; the centre of the hyperbola is at the point $O_1(0.5; 2)$, the branches of the hyperbola are located in the first and the third quadrant between its asymptotes $x - 0.5 = 0$, $y - 2 = 0$.

183. Reduce to the canonical form the equations of the parabolas: (1) $y = 4x - 2x^2$; (2) $y = -x^2 + 2x + 2$; (3) $x = -4y^2 + y$; (4) $x = y^2 + 4y + 5$.

184. Transform the equations of the hyperbolas to the form $x'y' = m$: (1) $y = 2x/(4x - 1)$; (2) $y = (2x + 3)/(3x - 2)$; (3) $y = (10x + 2)/(5x + 4)$; (4) $y = (4x + 3)/(2x + 1)$.

1.4.3. A quintic equation of a quadratic curve. The second-degree equation of the form

$$Ax^2 + Cy^2 + 2Dx + 2Ey + F = 0$$

(which does not contain the term xy with the product of coordinates) is called a *quintic equation of a quadratic curve*. It specifies, on the xOy plane, an ellipse, a hyperbola or parabola (with the possible cases of disintegration and degeneration of these curves) with the symmetry axes parallel to the axes of coordinates, depending on the sign of the product of the coefficients A and C.

1. Suppose $AC > 0$; then the curve specified by this equation is an ellipse (a real, imaginary or a point ellipse); at $A = C$ the ellipse becomes a circle.

2. Suppose $AC < 0$; then the corresponding curve is a hyperbola, which can degenerate into two intersecting straight lines if the left-hand side of the equation disintegrates into a product of two linear multipliers:

$$Ax^2 + Cy^2 + 2Dx + 2Ey + F = (a_1x + b_1y + c_1)(a_2x + b_2y + c_2).$$

3. Suppose $AC = 0$ (i.e. either $A = 0$, $C \neq 0$, or $C = 0$, $A \neq 0$); then the equation specifies a parabola which can degenerate into two parallel straight lines (real distinct, real merged or imaginary) if the left-hand side of the equation does not contain either x or y (that is, if the equation has the form $Ax^2 + 2Dx + F = 0$ or $Cy^2 + 2Ey + F = 0$).

The form of the curve and its location on the plane can be easily established by reducing the equation to the form $A(x - x_0)^2 + C(y - y_0)^2 = f$ (in the case $AC > 0$ or $AC < 0$); the form of the equation obtained shows the cases of disintegration or degeneration of an ellipse or a hyperbola.

In the case of nondegenerate curves, the displacement of the origin to the point $O_1(x_0; y_0)$ can reduce the equation of an ellipse or a hyperbola to the canonical form.

The case $AC = 0$ was treated in detail in the previous section since the equation of a nondegenerate parabola may be written here in the form $y = a_1x^2 + b_1x + c$ or $x = a_1y^2 + b_1y + c_1$.

185. What line is specified by the equation $4x^2 + 9y^2 - 8x - 36y + 4 = 0$?

Solution. Let us transform the equation as follows:

$$4(x^2 - 2x) + 9(y^2 - 4y) = -4; \quad 4(x^2 - 2x + 1 - 1) + 9(y^2 - 4y + 4 - 4) = -4;$$
$$4(x - 1)^2 + 9(y - 2)^2 = -4 + 4 + 36; \quad 4(x - 1)^2 + 9(y - 2)^2 = 36.$$

Taking the point $O'(1; 2)$ as the new origin, we perform a parallel translation of the coordinate axes. Next we make use of the formulas for the transformation of coordinates: $x = x' + 1$, $y = y' + 2$. With respect to the new axes, the equation of the curve will assume the form

$$4x'^2 + 9y'^2 = 36, \quad \text{or} \quad \frac{x'^2}{9} + \frac{y'^2}{4} = 1.$$

Thus we see that the given curve is an ellipse.

186. What line is specified by the equation $x^2 - 9y^2 + 2x + 36y - 44 = 0$?

Solution. Let us transform the given equation as follows:

$$(x^2 + 2x + 1 - 1) - 9(y^2 - 4y + 4 - 4) = 44;$$
$$(x + 1)^2 - 9(y - 2)^2 = 44 + 1 - 36, \ (x + 1)^2 - 9(y - 2)^2 = 9.$$

Taking the point $O'(-1; 2)$ as the new origin, we perform a parallel translation of the coordinate axes. The formulas for transformation of the coordinates have the form $x = x' - 1$, $y = y' + 2$. The transformation of the coordinates results in the equation

$$x'^2 - 9y'^2 = 9, \quad \text{or} \quad \frac{x'^2}{9} - y'^2 = 1.$$

We see that the curve is a hyperbola. The asymptotes to this hyperbola about the new axes are the lines $y' = (\pm 1/3)x'$.

Determine what curves are specified by the following equations. Make the drawings.

187. $36x^2 + 36y^2 - 36x - 24y - 23 = 0.$
188. $16x^2 + 25y^2 - 32x + 50y - 359 = 0.$

189. $\frac{1}{4}x^2 - \frac{1}{9}y^2 - x + \frac{2}{3}y - 1 = 0.$

190. $x^2 + 4y^2 - 4x - 8y + 8 = 0.$
191. $x^2 + 4y^2 + 8y + 5 = 0.$
192. $x^2 - y^2 - 6x + 10 = 0.$
193. $2x^2 - 4x + 2y - 3 = 0.$
194. $x^2 - 6x + 8 = 0.$
195. $x^2 + 2x + 5 = 0.$

1.4.4. Reducing a general equation of a quadratic curve to the canonical form. If a quadratic curve is specified by the equation

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0,$$

then, employing the transformation consisting in the rotation of the coordinate axes with the use of the formulas $x = x' \cos\alpha - y' \sin\alpha, y = x' \sin\alpha + y' \cos\alpha$, it is necessary to make a proper choice of α and delete from the equation the term with the product of coordinates.

Further transformations have been considered in the previous section.

The case of disintegration of a quadratic curve into two straight lines can be easily established from the original equation in the following way: considering the equation as a quadratic one with respect to y (assuming that the coefficient in y^2 is nonzero), we solve it with respect to y; if in this case the integrand turns to be a perfect square of a certain binomial $ax + b$ we shall be able to extract the root and obtain two values for y: $y_1 = k_1x + b_1$; $y_2 = k_2x + b_2$. And this shows that the curve disintegrates into two straight lines.

The given equation can also be solved with respect to x. If $A = C = 0$ in a general equation of a quadratic curve (naturally $B \neq 0$), the indicated equation specifies a pair of straight lines if and only if $B/D = 2E/T$. In that case the left-hand side of the equation can be factorized into linear components.

196. Show that the equation $9x^2 + 24xy + 16y^2 - 25 = 0$ specifies a set of two straight lines.

Solution. Let us rewrite the equation in the form $(3x + 4y)^2 - 25 = 0$. Factorizing the left-hand side of the equation, we obtain $(3x + 4y + 5)(3x + 4y -$

$-5) = 0$. Thus, the given equation specifies the lines $3x + 4y + 5 = 0$ and $3x + 4y - 5 = 0$.

197. Show that the equation $3x^2 + 8xy - 3y^2 - 14x - 2y + 8 = 0$ specifies a set of two straight lines.

Solution. We rewrite the equation in the form

$$3y^2 - 2(4x - 1)y - (3x^2 - 14x + 8) = 0.$$

Next we solve the equation with respect to y:

$$y = \frac{4x - 1 \pm \sqrt{(4x - 1)^2 + (9x^2 - 42x + 24)}}{3} \text{ or } y = \frac{4x - 1 \pm (5x - 5)}{3}.$$

We obtain the equations of the straight lines $y = 3x - 2$ and $y = (-x + 4)/3$. These equations can be written in the form $3x - y - 2 = 0$, $x + 3y - 4 = 0$.

198. What line is specified by the equation $xy + 2x - 4y - 8 = 0$?

Solution. Let us write the equation in the form

$$x(y + 2) - 4(y + 2) = 0, \quad \text{or} \quad (x - 4)(y + 2) = 0.$$

Thus, the equation specifies two lines $x - 4 = 0$ and $y + 2 = 0$, one of which is parallel to the x-axis and the other to the y-axis.

199. Reduce the following equation to the canonical form:

$$5x^2 + 4xy + 8y^2 + 8x + 14y + 5 = 0.$$

Solution. 1. Let us transform this equation, making use of formulas (3) for the rotation of the coordinate axes. We have

$$5(x' \cos\alpha - y' \sin\alpha)^2 + 4(x'\cos\alpha - y' \sin\alpha)(x' \sin\alpha + y' \cos\alpha) + 8(x' \sin\alpha + y' \cos\alpha)^2 + 8(x' \cos\alpha - y' \sin\alpha) + 14(x' \sin\alpha + y' \cos\alpha) + 5 = 0,$$

or

$$(5 \cos^2\alpha + 4 \sin\alpha \cos\alpha + 8 \sin^2\alpha)x'^2 + (5 \sin^2\alpha - 4 \sin\alpha \cos\alpha + 8 \cos^2\alpha)y'^2 + [6 \sin\alpha \cos\alpha + 4(\cos^2\alpha - \sin^2\alpha)]x'y' + (8 \cos\alpha + 14 \sin\alpha)x' + (14 \cos\alpha - 8 \sin\alpha)y' + 5 = 0.$$

Let us now find α from the condition $4(\cos^2\alpha - \sin^2\alpha) + 6 \sin\alpha \cos\alpha = 0$, that is, equate to zero the coefficient in $x'y'$. We get the equation $2 \tan^2\alpha - 3 \tan\alpha - 2 = 0$. Hence we have $\tan\alpha_1 = 2$, $\tan\alpha_2 = -1/2$.

Note that these values of $\tan\alpha$ correspond to two mutually perpendicular directions. Therefore, taking $\tan\alpha = 2$ instead of $\tan\alpha = -1/2$, we only make the axes x' and y' exchange their roles (Fig. 15).

Assume that $\tan\alpha = 2$, then $\sin\alpha = \pm 2/\sqrt{5}$, $\cos\alpha = \pm 1/\sqrt{5}$; if we take the positive values of $\sin\alpha$ and $\cos\alpha$, the equation assumes the form

$$9x'^2 + 4y'^2 + \frac{36}{\sqrt{5}}x' - \frac{2}{\sqrt{5}}y' + 5 = 0,$$

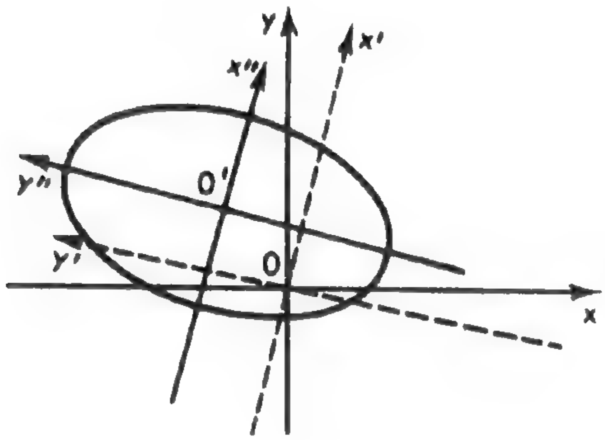

Fig. 15

or

$$9\left(x'^2 + \frac{4}{\sqrt{5}}x'\right) + 4\left(y'^2 - \frac{1}{2\sqrt{5}}y'\right) = -5.$$

2. Let us complete the expressions in parentheses to perfect squares:

$$9\left(x' + \frac{2}{\sqrt{5}}\right)^2 + 4\left(y' - \frac{1}{4\sqrt{5}}\right)^2 = \frac{36}{5} + \frac{1}{20} - 5,$$

or

$$9\left(x' + \frac{2}{\sqrt{5}}\right)^2 + 4\left(y' - \frac{1}{4\sqrt{5}}\right)^2 = \frac{9}{4}.$$

Taking the point $O'(-2/\sqrt{5}; 1/(4\sqrt{5})$ as the new origin, we apply the formulas for transformation of coordinates $x' = x'' - 2/\sqrt{5}, y' = y'' + 1/(4\sqrt{5})$; we get

$$9x''^2 + 4y''^2 = \frac{9}{4} \quad \text{or} \quad \frac{x''^2}{1/4} + \frac{y''^2}{9/16} = 1$$

(equation of an ellipse).

200. Reduce the following equation to the canonical form:

$$6xy + 8y^2 - 12x - 26y + 11 = 0.$$

Solution. 1. Transform the equation making use of formulas (3) for the rotation of the coordinate axes

$$6(x'\cos\alpha - y'\sin\alpha)(x'\sin\alpha + y'\cos\alpha)$$
$$+ 8(x'\sin\alpha + y'\cos\alpha)^2 - 12(x'\cos\alpha - y'\sin\alpha)$$
$$- 26(x'\sin\alpha + y'\cos\alpha) + 11 = 0,$$

or

$$(6\sin\alpha\cos\alpha + 8\sin^2\alpha)x'^2 + (8\cos^2\alpha - 6\sin\alpha\cos\alpha)y'^2$$

$$+ [16 \sin\alpha \cos\alpha + 6(\cos^2\alpha - \sin^2\alpha)]x'y'$$
$$- (12 \cos\alpha + 26 \sin\alpha)x' - (26 \cos\alpha - 12 \sin\alpha)y' + 11 = 0.$$

Equating the coefficient in $x'y'$ to zero yields

$16 \sin\alpha \cos\alpha + 6(\cos^2\alpha - \sin^2\alpha) = 0$, or $3 \tan^2\alpha - 8 \tan\alpha - 3 = 0$.

Hence, $\tan\alpha_1 = 3$, $\tan\alpha_2 = -1/3$; assume $\tan\alpha = 3$ and get $\sin\alpha = \pm 3/\sqrt{10}$, $\cos\alpha = \pm 1/\sqrt{10}$; then take the positive values of $\sin\alpha$ and $\cos\alpha$. The equation assumes the form

$$9x'^2 - y'^2 - 9\sqrt{10}x' + \sqrt{10}y' + 11 = 0, \quad \text{or} \quad 9(x'^2 - \sqrt{10}x') - (y'^2 - \sqrt{10}y') = -11.$$

2. Complete the expressions in parentheses to perfect squares:

$$9\left(x' - \frac{\sqrt{10}}{2}\right)^2 - \left(y' - \frac{\sqrt{10}}{2}\right)^2 = \frac{45}{2} - \frac{5}{2} = 11.$$

or

$$9\left(x' - \frac{\sqrt{10}}{2}\right)^2 - \left(y' - \frac{\sqrt{10}}{2}\right)^2 = 9.$$

Taking the point $O'(\sqrt{10}/2; \sqrt{10}/2)$ as a new origin, apply the formulas for transformation of coordinates $x' = x'' + \sqrt{10}/2$, $y' = y'' + \sqrt{10}/2$. The result is

$$9x''^2 - y''^2 = 9, \quad \text{or} \quad x''^2 - \frac{y''^2}{9} = 1$$

(equation of a hyperbola).

201. Reduce the following equation to the canonical form:

$$x^2 - 2xy + y^2 - 10x - 6y + 25 = 0.$$

Solution. 1. Let us transform the equation with the aid of the formulas for the rotation of the axes:

$$(x'\cos\alpha - y'\sin\alpha)^2 - 2(x'\cos\alpha - y'\sin\alpha)(x'\sin\alpha + y'\cos\alpha)$$
$$+ (x'\sin\alpha + y'\cos\alpha)^2 - 10(x'\cos\alpha - y'\sin\alpha) - 6(x'\sin\alpha + y'\cos\alpha) + 25 = 0$$

or

$$(\cos^2\alpha - 2 \sin\alpha \cos\alpha + \sin^2\alpha)x'^2 + (\sin^2\alpha + 2 \sin\alpha \cos\alpha + \cos^2\alpha)y'^2 + 2(\sin^2\alpha - \cos^2\alpha)x'y' - (10 \cos\alpha + 6 \sin\alpha)x' + (10 \sin\alpha - 6 \cos\alpha)y' + 25 = 0.$$

Equating the coefficient in the product $x'y'$ to zero, we get

$2(\sin^2\alpha - \cos^2\alpha) = 0$, whence $\tan^2\alpha = 1$, i.e. $\tan\alpha_1 = 1$, $\tan\alpha_2 = -1$. We take $\tan\alpha = 1$, whence we have $\alpha = \pi/4$ and $\sin\alpha = 1/\sqrt{2}$, $\cos\alpha = 1/\sqrt{2}$. Then the equation assumes the form

$$2y'^2 - 8\sqrt{2}x' + 2\sqrt{2}y' + 25 = 0, \quad \text{or} \quad 2(y'^2 + \sqrt{2}y') - 8\sqrt{2}x' + 25 = 0.$$

2. We complete the expression in parentheses to a perfect square:

$$2\left(y' + \frac{\sqrt{2}}{2}\right)^2 = 8\sqrt{2}x' - 24$$

or

$$\left(y' + \frac{\sqrt{2}}{2}\right)^2 = 4\sqrt{2}\left(x' - \frac{3}{\sqrt{2}}\right).$$

Taking the point $O'(3/\sqrt{2}; -\sqrt{2}/2)$ as a new origin, we apply the formulas for transformation of coordinates $x' = x'' - 3/\sqrt{2}$, $y' = y'' + \sqrt{2}/2$ and get

$$y''^2 = 4\sqrt{2}x''$$

(equation of a parabola).

Show that the following equations specify curves disintegrating into a pair of straight lines and find the equations of these lines:

202. $25x^2 + 10xy + y^2 - 1 = 0$.
203. $x^2 + 2xy + y^2 + 2x + 2y + 1 = 0$.
204. $8x^2 - 18xy + 9y^2 + 2x - 1 = 0$.

Reduce the equations of the following curves to the canonical form:

205. $14x^2 + 24xy + 21y^2 - 4x + 18y - 139 = 0$.
206. $4xy + 3y^2 + 16x + 12y - 36 = 0$.
207. $9x^2 - 24xy + 16y^2 - 20x + 110y - 50 = 0$.

1.5. Second- and Third-Order Determinants and Systems of Linear Equations in Two and Three Unknowns

1.5.1. Second-order determinants and systems of linear equations. The *second-order determinant* corresponding to the table of elements $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ is specified by the equality

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

The system of two linear equations in two unknowns

$$\begin{cases} a_1x + b_1y = c_1, \\ a_2x + b_2y = c_2, \end{cases}$$

in the case its determinant is $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$, has a unique solution which can be found from the formulas

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}; \quad y = \frac{D_y}{D} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \tag{1}$$

(*Cramer's rule*).

If the determinant $D = 0$, then the system is either inconsistent (when $D_x \neq 0$ and $D_y \neq 0$), or indeterminate (when $D_x = D_y = 0$). In the last case the system can be reduced to one equation (say, the first), the other equation being the corollary of the first one.

The condition of inconsistency of the system can be written in the form $a_1/a_2 = b_1/b_2 \neq c_1/c_2$, and the condition of indeterminacy, in the form $a_1/a_2 = b_1/b_2 = c_1/c_2$.

A linear equation is said to be *homogeneous* if the constant term of the equation is zero.

Let us consider a system of two homogeneous linear equations in three unknowns:

$$\begin{cases} a_1x + b_1y + c_1z = 0, \\ a_2x + b_2y + c_2z = 0. \end{cases}$$

1. If $a_1/a_2 = b_1/b_2 = c_1/c_2$, then the system reduces to one equation (say, the first) in which one of the unknowns can be expressed in terms of the other two whose values remain arbitrary.

2. If the condition $a_1/a_2 = b_1/b_2 = c_1/c_2$ is not satisfied, then the solution of the system can be found from the formulas

$$x = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \cdot t, \quad y = -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \cdot t, \quad z = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \cdot t, \tag{2}$$

where t can assume any values. These solutions can also be written as a proportion:

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{y}{-\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = t.$$

But when the solutions are written in this form, it should be remembered that if one of the denominators vanishes, the corresponding numerator should be equated to zero.

208. Solve the following system of equations:

$$\begin{cases} (a+b)x - (a-b)y = 4ab, \\ (a-b)x + (a+b)y = 2(a^2 - b^2). \end{cases}$$

Solution. We find the determinant D of the system and the determinants D_x and D_y entering into the numerators of formulas (1):

$$D = \begin{vmatrix} a+b & -(a-b) \\ a-b & a+b \end{vmatrix} = (a+b)^2 + (a-b)^2 = 2(a^2+b^2),$$

$$D_x = \begin{vmatrix} 4ab & -(a-b) \\ 2(a^2-b^2) & a+b \end{vmatrix} = 4a^2b + 4ab^2 + 2a^3 - 2a^2b$$

$$-2ab^2 + 2b^3 = 2(a^3 + a^2b + ab^2 + b^3) = 2(a^2+b^2)(a+b),$$

$$D_y = \begin{vmatrix} a+b & 4ab \\ a-b & 2(a^2-b^2) \end{vmatrix} = 2a^3 + 2a^2b - 2ab^2 - 2b^3$$

$$-4a^2b + 4ab^2 = 2(a^3 - a^2b + ab^2 - b^3) = 2(a^2+b^2)(a-b).$$

Hence $x = D_x/D = a + b$, $y = D_y/D = a - b$.

209. Solve the following system of homogeneous linear equations:

$$\begin{cases} 3x + 4y + 5z = 0, \\ x + 2y - 3z = 0. \end{cases}$$

Solution. Using formulas (2), we find

$$x = \begin{vmatrix} 4 & 5 \\ 2 & -3 \end{vmatrix} \cdot t = -22t, \quad y = -\begin{vmatrix} 3 & 5 \\ 1 & -3 \end{vmatrix} \cdot t = 14t, \quad z = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} \cdot t = 2t,$$

where t can assume any values.

Solve the following systems of equations:

210. $\begin{cases} 5x - 3y = 1, \\ x + 11y = 6. \end{cases}$ **211.** $\begin{cases} 2x + y = 1/5, \\ 4x + 2y = 1/3. \end{cases}$

212. $\begin{cases} ax - by = a^2 + b^2, \\ bx + ay = a^2 + b^2. \end{cases}$ **213.** $\begin{cases} 3x + 2y = 1/6, \\ 9x + 6y = 1/2. \end{cases}$

214. $\begin{cases} x - 2y + z = 0, \\ 3x - 5y + 2z = 0. \end{cases}$ **215.** $\begin{cases} x\cos\alpha - y\sin\alpha = \cos 2\alpha, \\ x\sin\alpha + y\cos\alpha = \sin 2\alpha. \end{cases}$

216. $\begin{cases} a^2x - 2(a^2 + b^2)y + b^2z = 0, \\ 2x + 2y - 3z = 0. \end{cases}$

1.5.2. Third-order determinants and systems of linear equations. The *determinant of the third order* corresponding to the table of elements $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ is specified by the equality

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

A *minor* (*complementary minor*) of a given element in a third-order determinant is a second-order determinant which is obtained by striking out, in the original determinant, the row and the column in which the element lies. An *algebraic adjunct* (a *signed minor* or a *cofactor*) of an element is its minor multiplied by $(-1)^k$, where k is the sum of the position numbers of the row and the column containing the given element.

The sign which is then given to the minor of the corresponding element of the determinant is defined by the following table:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}.$$

The right-hand side of the above equality specifying a third-order determinant includes the sum of the products of the first row elements of the determinant by their cofactors.

The following general **theorem** holds true: *a third-order determinant is equal to the sum of the products of the elements of its any row or column by their cofactors.* This theorem makes it possible to calculate the value of a determinant by expanding it with respect to the elements of its any row or column.

Note that the following **theorem** is also true: *the sum of the products of the elements of any row (column) of a determinant by the cofactors of the elements of another row (column) is equal to zero.*

Properties of Determinants

1°. *A determinant does not change if its rows are replaced by its columns and the columns by the corresponding rows.*

2°. *A common factor of the elements of any row (or column) can be taken outside the determinant.*

3°. *If the elements of one row (column) of a determinant are respectively equal to the elements of another row (column), then the determinant is equal to zero.*

4°. *A determinant changes sign upon an interchange of two rows (columns).*

5°. *A determinant does not change as a result of an addition to the elements of one row (column) of the respective elements of another row (column) multiplied by the same number* (a theorem on a linear combination of the parallel rows of a determinant).

A solution of a system of three linear equations in three unknowns

$$\begin{cases} a_1x + b_1y + c_1z = d_1, \\ a_2x + b_2y + c_2z = d_2, \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

can be found using Cramer's rule

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad z = \frac{D_z}{D}, \tag{1}$$

where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \quad D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

In this case it is assumed that $D \neq 0$ (if $D = 0$, the original system is either indeterminate or inconsistent).

If a system is homogeneous, that is, has the form

$$\begin{cases} a_1x + b_1y + c_1z = 0, \\ a_2x + b_2y + c_2z = 0, \\ a_3x + b_3y + c_3z = 0, \end{cases}$$

and its determinant is nonzero, then it has a unique solution $x = 0, y = 0, z = 0$.

If, now, the determinant of a homogeneous system is equal to zero, the system reduces either to two independent equations (the third equation being their corollary), or to one equation (the other two being its corollaries). The first case obtains if there is at least one nonzero minor among the minors of the determinant of a homogeneous system, the second case obtains if all the minors of that determinant are equal to zero.

In either case (see 1.5.1) the homogeneous system possesses an infinite number of solutions.

217. Calculate the following third-order determinant:

$$\begin{vmatrix} 5 & 3 & 2 \\ -1 & 2 & 4 \\ 7 & 3 & 6 \end{vmatrix}.$$

Solution. Expanding the determinant with respect to the elements of the first row, we obtain

$$\begin{vmatrix} 5 & 3 & 2 \\ -1 & 2 & 4 \\ 7 & 3 & 6 \end{vmatrix} = 5\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} - 3\begin{vmatrix} -1 & 4 \\ 7 & 6 \end{vmatrix} + 2\begin{vmatrix} -1 & 2 \\ 7 & 3 \end{vmatrix} = 5\cdot 0 - 3(-34) + $$
$$+ 2(-17) = 68.$$

218. Calculate the same determinant proceeding from the theorem on a linear combination of the elements of the rows (columns).

Solution. To the elements of the first row we add the respective elements of the second row multiplied by 5, and to the elements of the third row, the respective elements of the second row multiplied by 7:

$$\begin{vmatrix} 5 & 3 & 2 \\ -1 & 2 & 4 \\ 7 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 0 & 13 & 22 \\ -1 & 2 & 4 \\ 0 & 17 & 34 \end{vmatrix}.$$

Expanding the determinant with respect to the elements of the first row, we get

$$\begin{vmatrix} 0 & 13 & 22 \\ -1 & 2 & 4 \\ 0 & 17 & 34 \end{vmatrix} = 0 \cdot \begin{vmatrix} 2 & 4 \\ 17 & 34 \end{vmatrix} + 1 \cdot \begin{vmatrix} 13 & 22 \\ 17 & 34 \end{vmatrix} + 0 \cdot \begin{vmatrix} 13 & 22 \\ 2 & 4 \end{vmatrix} = 13 \cdot 34 -$$
$$- 17 \cdot 22 = 68.$$

219. Solve the following system of equations:

$$\begin{cases} x + 2y + z = 8, \\ 3x + 2y + z = 10, \\ 4x + 3y - 2z = 4. \end{cases}$$

Solution. We find from formulas (1)

$$x = \frac{\begin{vmatrix} 8 & 2 & 1 \\ 10 & 2 & 1 \\ 4 & 3 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 4 & 3 & -2 \end{vmatrix}} = \frac{8\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} - 2\begin{vmatrix} 10 & 1 \\ 4 & -2 \end{vmatrix} + 1\cdot\begin{vmatrix} 10 & 2 \\ 4 & 3 \end{vmatrix}}{1\cdot\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} + 1\cdot\begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix}} = \frac{14}{14} = 1,$$

$$y = \frac{\begin{vmatrix} 1 & 8 & 1 \\ 3 & 10 & 1 \\ 4 & 4 & -2 \end{vmatrix}}{14} = \frac{1 \cdot \begin{vmatrix} 10 & 1 \\ 4 & -2 \end{vmatrix} - 8 \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 3 & 10 \\ 4 & 4 \end{vmatrix}}{14} = \frac{28}{14} = 2,$$

$$z = \frac{\begin{vmatrix} 1 & 2 & 8 \\ 3 & 2 & 10 \\ 4 & 3 & 4 \end{vmatrix}}{14} = \frac{1 \cdot \begin{vmatrix} 2 & 10 \\ 3 & 4 \end{vmatrix} - 2 \begin{vmatrix} 3 & 10 \\ 4 & 4 \end{vmatrix} + 8 \begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix}}{14} = \frac{42}{14} = 3,$$

220. Solve the following system of homogeneous linear equations:

$$\begin{cases} 4x + y + z = 0, \\ x + 3y + z = 0, \\ x + y + 2z = 0. \end{cases}$$

Solution. Here $D = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{vmatrix}$. To calculate the determinant, we add to the elements of the first row the elements of the third row multiplied by -4, and to the elements of the second row, the elements of the third row multiplied by -1:

$$D = \begin{vmatrix} 0 & -3 & -7 \\ 0 & 2 & -1 \\ 1 & 1 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} -3 & -7 \\ 2 & -1 \end{vmatrix} = 17.$$

Since $D \neq 0$, the system possesses only a zero solution $x = y = z = 0$.

221. Solve the following system of equations:

$$\begin{cases} 3x + 2y - z = 0, \\ x + 2y + 9z = 0, \\ x + y + 2z = 0. \end{cases}$$

Solution. We have

$$D = \begin{vmatrix} 3 & 2 & -1 \\ 1 & 2 & 9 \\ 1 & 1 & 2 \end{vmatrix} = 3 \cdot \begin{vmatrix} 2 & 9 \\ 1 & 2 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 9 \\ 1 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} =$$
$$= -15 + 14 + 1 = 0.$$

Consequently, the system possesses nonzero solutions. We solve the system of the first two equations (the third equation is their corollary):

$$\begin{cases} 3x + 2y - z = 0, \\ x + 2y + 9z = 0. \end{cases}$$

From this we get, from formulas (2) of 1.5.1,

$$x = \begin{vmatrix} 2 & -1 \\ 2 & 9 \end{vmatrix} \cdot t = 20t, \quad y = -\begin{vmatrix} 3 & -1 \\ 1 & 9 \end{vmatrix} \cdot t = -28t, \quad z = \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} \cdot t = 4t.$$

222. Compute the determinant

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 7 & 2 \\ 2 & 3 & -7 \end{vmatrix},$$

by expanding it with respect to the elements of the third row.

223. Compute the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 9 & 16 \end{vmatrix},$$

using the theorem on a linear combination of rows (columns).

224. Compute the determinant

$$\begin{vmatrix} 2 & 3 & 4 \\ 2 & a+3 & b+4 \\ 2 & c+3 & d+4 \end{vmatrix}.$$

Solve the following systems of equations:

225. $\begin{cases} 5x - y - z = 0, \\ x + 2y + 3z = 14, \\ 4x + 3y + 2z = 16. \end{cases}$ **226.** $\begin{cases} x + 3y - 6z = 12, \\ 3x + 2y + 5z = -10, \\ 2x + 5y - 3z = 6. \end{cases}$

227. $\begin{cases} -5x + y + z = 0, \\ x - 6y + z = 0, \\ x + y - 7z = 0. \end{cases}$ **228.** $\begin{cases} x + y + z = 0, \\ 3x + 6y + 5z = 0, \\ x + 4y + 3z = 0. \end{cases}$

229. $\begin{cases} ax + by + cz = a - b, \\ bx + cy + az = b - c, \\ cx + ay + bz = c - a, \text{ if } a + b + c \neq 0. \end{cases}$

230. $\begin{cases} ax + by + (a + b)z = 0, \\ bx + ay + (a + b)z = 0, \\ x + y + 2z = 0. \end{cases}$

Chapter 2

Elements of Vector Algebra

2.1. Rectangular Coordinates in Space

If a rectangular Cartesian system of coordinates $Oxyz$ is given in space, then a point M of the space having the coordinates x (abscissa), y (ordinate) and z (z-coordinate) is designated as $M(x; y; z)$.

A distance between two points $A(x_1; y_1; z_1)$ and $B(x_2; y_2; z_2)$ can be determined from the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{1}$$

In particular, the distance of the point $M(x; y; z)$ from the origin O can be found from the formula

$$d = \sqrt{x^2 + y^2 + z^2}. \tag{2}$$

If a line segment whose end points are the points $A(x_1; y_1; z_1)$ and $B(x_2; y_2; z_2)$ is divided by the point $C(x; y; z)$ in the ratio λ (see 1.1), then the coordinates of the point M can be determined by the formulas

$$x = \frac{x_1 + \lambda x_2}{1 + \lambda}; \quad y = \frac{y_1 + \lambda y_2}{1 + \lambda}; \quad z = \frac{z_1 + \lambda z_2}{1 + \lambda}. \tag{3}$$

In particular, the coordinates of the midpoint of the segment can be found from the formulas

$$x = \frac{x_1 + x_2}{2}; \quad y = \frac{y_1 + y_2}{2}; \quad z = \frac{z_1 + z_2}{2}. \tag{4}$$

231. Given the points $M_1(2; 4; -2)$ and $M_2(-2; 4; 2)$. On the line M_1M_2 find a point M which divides the segment M_1M_2 in the ratio $\lambda = 3$.

Solution. We apply the formulas for division of a segment in a given ratio:

$$x_M = \frac{x_1 + \lambda x_2}{1 + \lambda} = \frac{2 + 3(-2)}{1 + 3} = -1,$$

$$y_M = \frac{y_1 + \lambda y_2}{1 + \lambda} = \frac{4 + 3 \cdot 4}{1 + 3} = 4,$$

$$z_M = \frac{z_1 + \lambda z_2}{1 + \lambda} = \frac{-2 + 3 \cdot 2}{1 + 3} = 1.$$

Consequently, the point in question is $M(-1; 4; 1)$.

232. Given the triangle with the vertices $A(1; 1; 1)$, $B(5; 1; -2)$, $C(7; 9; 1)$. Find the coordinates of the point D of intersection of the bisector of the angle A and the side CB.

Solution. Let us find the lengths of the sides of the triangle forming the angle A:

$$|AC| = \sqrt{(x_C - x_A)^2 + (y_C - y_A)^2 + (z_C - z_A)^2}$$
$$= \sqrt{(7-1)^2 + (9-1)^2 + (1-1)^2} = 10;$$
$$|AB| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$
$$= \sqrt{(5-1)^2 + (1-1)^2 + (-2-1)^2} = 5.$$

Consequently, $|CD| : |DB| = 10 : 5 = 2$, since the bisector divides the side CB into parts proportional to the adjacent sides. Thus we have

$$x_D = \frac{x_C + \lambda x_B}{1+\lambda} = \frac{7 + 2\cdot 5}{1+2} = \frac{17}{3},$$
$$y_D = \frac{y_C + \lambda y_B}{1+\lambda} = \frac{9 + 2\cdot 1}{1+2} = \frac{11}{3},$$
$$z_D = \frac{z_C + \lambda z_B}{1+\lambda} = \frac{1 + 2(-2)}{1+2} = -1;$$

the desired point is $D(17/3; 11/3; -1)$.

233. Find the point on the x-axis which is equidistant from the points $A(2; -4; 5)$ and $B(-3; 2; 7)$.

Solution. Suppose M is the desired point. The equality $|AM| = |MB|$ should be fulfilled for it. Since the point is on the x-axis, its coordinates are $(x; 0; 0)$ and, therefore, we have

$$|AM| = \sqrt{(x-2)^2 + (-4)^2 + 5^2}, \quad |MB| = \sqrt{(x+3)^2 + 2^2 + 7^2}.$$

From this we obtain, after squaring,

$$(x-2)^2 + 41 = (x+3)^2 + 53, \quad \text{or} \quad 10x = -17, \quad \text{i.e.} \quad x = 1.7.$$

Thus, the desired point $M(-1.7; 0; 0)$.

234. Given the points $A(3; 3; 3)$ and $B(-1; 5; 7)$. Find the coordinates of the points C and D dividing the line segment AB into three congruent parts.

235. Given the triangle with the vertices $A(1; 2; 3)$, $B(7; 10; 3)$, $C(-1, 3; 1)$. Show that the angle A is an obtuse one.

236. Determine the coordinates of the centre of gravity of a triangle with the vertices $A(2; 3; 4)$, $B(3; 1; 2)$, $C(4; -1; 3)$.

237. In what ratio does a point M, equidistant from the points $A(3; 1; 4)$ and $B(-4; 5; 3)$, divide the y-intercept from the origin to the point $C(0; 6; 0)$?

238. Find a point on the z-axis equidistant from the points $M_1(2; 4; 1)$ and $M_2(-3; 2; 5)$.

239. Find the point on the xOy plane equidistant from the points $A(1; -1; 5)$, $B(3; 4; 4)$ and $C(4; 6; 1)$.

2.2. Vectors and the Simplest Operations on Them

A free vector $\mathbf{a}$ (that is, a vector that can be translated to any point of space without a change in its length or direction), given in the coordinate space $Oxyz$, can be represented in the form

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}.$$

Such a representation of the vector $\mathbf{a}$ is called its *resolution along the axes of coordinates* or *resolution with respect to unit vectors.*

Here a_x, a_y, a_z are the projections of the vector $\mathbf{a}$ on the corresponding coordinate axes (they are called the *coordinates* of the vector $\mathbf{a}$), $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ being the unit vectors of these axes (unit vectors whose directions coincide with the positive direction of the corresponding axis).

The vectors $a_x\mathbf{i}$, $a_y\mathbf{j}$ and $a_z\mathbf{k}$ whose sum represents the vector $\mathbf{a}$ are called the *components* of the vector $\mathbf{a}$ along the coordinate axes.

The numerical length (absolute value) of the vector $\mathbf{a}$ is designated as a or $|\mathbf{a}|$ and can be determined by the formula

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}.$$

The direction of the vector $\mathbf{a}$ is determined by the angles α, β and γ which it makes with the coordinate axes Ox, Oy and Oz. The cosines of these angles (the so-called *direction cosines of a vector*) can be found from the formulas

$$\cos\alpha = \frac{a_x}{a} = \frac{a_x}{\sqrt{a_x^2 + a_y^2 + a_z^2}}; \quad \cos\beta = \frac{a_y}{a}; \quad \cos\gamma = \frac{a_z}{a}.$$

The direction cosines of a vector are related by the equation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

If the vectors $\mathbf{a}$ and $\mathbf{b}$ are given by their resolution with respect to the unit vectors, then their sum and difference are determined by the formulas

$$\mathbf{a} + \mathbf{b} = (a_x + b_x)\mathbf{i} + (a_y + b_y)\mathbf{j} + (a_z + b_z)\mathbf{k},$$
$$\mathbf{a} - \mathbf{b} = (a_x - b_x)\mathbf{i} + (a_y - b_y)\mathbf{j} + (a_z - b_z)\mathbf{k}.$$

Recall that the **sum** of the vectors $\mathbf{a}$ and $\mathbf{b}$ whose origins coincide is usually represented by a vector with the same origin, which coincides with the diagonal of a parallelogram whose sides are the vectors $\mathbf{a}$ and $\mathbf{b}$. The **difference** $\mathbf{a} - \mathbf{b}$ of these vectors is represented by a vector which coincides with the second diagonal of the same parallelogram, the origin of the vector being located at the terminus of the vector $\mathbf{b}$ and the terminus, at the terminus of the vector $\mathbf{a}$ (Fig. 16).

The sum of any number of vectors can be found by a polygon rule (Fig. 17).

The product of the vector $\mathbf{a}$ by a scalar factor m can be determined from the formula

$$m\mathbf{a} = ma_x\mathbf{i} + ma_y\mathbf{j} + ma_z\mathbf{k}.$$

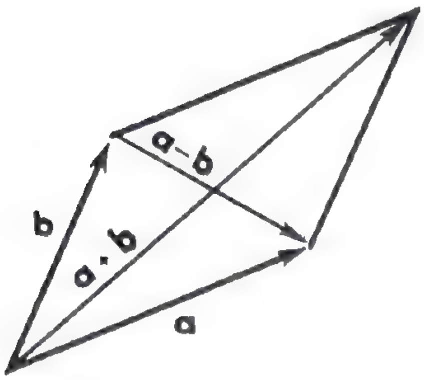

Fig. 16

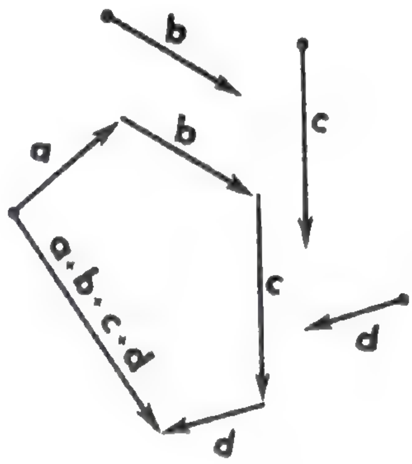

Fig. 17

Recall that the vectors $\mathbf{a}$ and $m\mathbf{a}$ are parallel (collinear) and point in the same direction if $m > 0$ and point in the opposite directions if $m < 0$.

In particular, if $m = 1/a$, then the vector $\mathbf{a}/a$ has a length equal to unity and a direction coinciding with that of the vector $\mathbf{a}$. This vector is called a *unit vector* of the vector $\mathbf{a}$ and is designated as $\mathbf{a}_0$.

Thus we have $\mathbf{a}_0 = \mathbf{a}/a$, or $\mathbf{a} = a\mathbf{a}_0$.

The vector $\overline{OM}$ which begins at the origin and terminates at the point $M(x; y; z)$ is called the *radius vector* of the point M and is denoted by $\mathbf{r}(M)$ or simply $\mathbf{r}$. Since its coordinates coincide with those of the point M, its resolution with respect to the unit vectors has the form

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

The vector $\overline{AB}$ originating at a point $A(x_1; y_1; z_1)$ and terminating at a point $B(x_2; y_2; z_2)$ can be written as $\overline{AB} = \mathbf{r}_2 - \mathbf{r}_1$, where $\mathbf{r}_2$ is the radius vector of the point B and $\mathbf{r}_1$ is the radius vector of the point A. Therefore, the resolution of the vector $\overline{AB}$ with respect to the unit vectors has the form

$$\overline{AB} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}.$$

Its length coincides with the distance between the points A and B:

$$|\overline{AB}| = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

By virtue of the formulas presented above, the direction of the vector $\overline{AB}$ is determined by the direction cosines:

$$\cos\alpha = \frac{x_2 - x_1}{d}; \quad \cos\beta = \frac{y_2 - y_1}{d}; \quad \cos\gamma = \frac{z_2 - z_1}{d}.$$

240. The side AB in the triangle ABC is divided by points M and N into three congruent parts: $|AM| = |MN| = |NB|$. Find the vector $\overline{CM}$ if $\overline{CA} = \mathbf{a}$, $\overline{CB} = \mathbf{b}$.

Solution. We have $\overline{AB} = \mathbf{b} - \mathbf{a}$. Consequently, $\overline{AM} = (\mathbf{b} - \mathbf{a})/3$. Since $\overline{CM} = \overline{CA} + \overline{AM}$, we have

$$\overline{CM} = \mathbf{a} + \frac{\mathbf{b} - \mathbf{a}}{3} = \frac{2\mathbf{a} + \mathbf{b}}{3}.$$

241. The straight line AM in the triangle ABC is the bisector of the angle BAC, the point M lying on the side BC. Find $\overline{AM}$ if $\overline{AB} = \mathbf{b}$, $\overline{AC} = \mathbf{c}$.

Solution. We have $\overline{BC} = \mathbf{c} - \mathbf{b}$. It follows from the property of the bisector of an interior angle of a triangle that $|BM| : |MC| = b : c$, i.e., $|BM| : |BC| = b : (b + c)$. Hence we obtain $\overline{BM} = \dfrac{b}{b+c}(\mathbf{c} - \mathbf{b})$. Since $\overline{AM} = \overline{AB} + \overline{BM}$, we have

$$\overline{AM} = \mathbf{b} + \frac{b}{b+c}(\mathbf{c} - \mathbf{b}) = \frac{b\mathbf{c} + c\mathbf{b}}{b+c}.$$

242. The radius vectors of the vertices of the triangle ABC are $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$. Find the radius vector of the intersection point of the medians of the triangle.

Solution. We have $\overline{BC} = \mathbf{r}_3 - \mathbf{r}_2$; $\overline{BD} = (\mathbf{r}_3 - \mathbf{r}_2)/2$ (D being the midpoint of the side BC); $\overline{AB} = \mathbf{r}_2 - \mathbf{r}_1$; $\overline{AD} = \overline{BD} + \overline{AB} = (\mathbf{r}_3 - \mathbf{r}_2)/2 + \mathbf{r}_2 - \mathbf{r}_1 = (\mathbf{r}_2 + \mathbf{r}_3 - 2\mathbf{r}_1)/2$; $\overline{AM} = (2/3)\overline{AD}$ (M being the point of intersection of the medians), therefore, $\overline{AM} = (\mathbf{r}_2 + \mathbf{r}_3 - 2\mathbf{r}_1)/3$. Thus we have

$$\mathbf{r} = \overline{OM} = \mathbf{r}_1 + \overline{AM} = (\mathbf{r}_2 + \mathbf{r}_3 - 2\mathbf{r}_1)/3 + \mathbf{r}_1, \quad \text{or} \quad \mathbf{r} = (\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3)/3.$$

243. Find the length of the vector $\mathbf{a} = 20\mathbf{i} + 30\mathbf{j} - 60\mathbf{k}$ and its direction cosines.

Solution. We have

$$a = \sqrt{20^2 + 30^2 + 60^2} = 70; \quad \cos\alpha = 20/70 = 2/7, \quad \cos\beta = 30/70 = 3/7,$$
$$\cos\gamma = -60/70 = -6/7.$$

244. Find the vector $\mathbf{a} = \overline{AB}$ if $A(1; 3; 2)$ and $B(5; 8; -1)$.

Solution. The projections of the vector $\overline{AB}$ on the coordinate axes are the differences of the corresponding coordinates of the points B and A: $a_x = 5 - 1 = 4$, $a_y = 8 - 3 = 5$, $a_z = -1 - 2 = -3$. Consequently, $\overline{AB} = 4\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$.

245. Given the triangle ABC. A point M is located on the side BC so that $|BM| : |MC| = \lambda$. Find $\overline{AM}$ if $\overline{AB} = \mathbf{b}$, $\overline{AC} = \mathbf{c}$.

246. Given $\overline{AB} = \mathbf{a} + 2\mathbf{b}$, $\overline{BC} = -4\mathbf{a} - \mathbf{b}$, $\overline{CD} = -5\mathbf{a} - 3\mathbf{b}$. Prove that $ABCD$ is a trapezoid.

247. Find the projections of the vector $\mathbf{a}$ on the coordinate axes if $\mathbf{a} = \overline{AB} + \overline{CD}$, $A(0; 0; 1)$, $B(3; 2; 1)$, $C(4; 6; 5)$ and $D(1; 6; 3)$.

248. Find the length of the vector $\mathbf{a} = m\mathbf{i} + (m + 1)\mathbf{j} + m(m + 1)\mathbf{k}$.

249. Given the radius vectors of the vertices of the triangle ABC: $\mathbf{r}_A = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{r}_B = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{r}_C = \mathbf{i} + 4\mathbf{j} + \mathbf{k}$. Show that the triangle ABC is equilateral.

250. Calculate the absolute value of the vector $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} - (1/5)(4\mathbf{i} + 8\mathbf{j} + 3\mathbf{k})$ and find its direction cosines.

251. Given the points $M_1(1; 2; 3)$ and $M_2(3; -4; 6)$. Find the length and the direction of the vector $\overline{M_1M_2}$.

252. Given the vector $\mathbf{a} = 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$. Find the vector $\mathbf{b}$ if $b = a$, $b_y = a_y$ and $b_x = 0$.

253. The radius vector of the point M forms an angle of $60°$ with the y-axis and an angle of $45°$ with the z-axis; its length is $z = 8$. Find the coordinates of the point M if its abscissa is negative.

2.3. Scalar and Vector Products. Mixed Product

2.3.1. A scalar product. The *scalar product* of two vectors $\mathbf{a}$ and $\mathbf{b}$ is a scalar equal to the product of the lengths of these vectors by the cosine of the angle φ between them:

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \varphi.$$

Properties of a Scalar Product

1°. $\mathbf{a} \cdot \mathbf{a} = a^2$, or $\mathbf{a}^2 = a^2$.
2°. $\mathbf{a} \cdot \mathbf{b} = 0$ if $\mathbf{a} = 0$, or $\mathbf{b} = 0$, or $\mathbf{a} \perp \mathbf{b}$.
3°. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (commutative law).
4°. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ (distributive law).
5°. $(m\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (m\mathbf{b}) = m(\mathbf{a} \cdot \mathbf{b})$ (associative law with respect to a scalar multiplier).

Scalar products of unit vectors of the coordinate axes are

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1, \quad \mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = 0.$$

Suppose the vectors $\mathbf{a}$ and $\mathbf{b}$ are specified by their coordinates: $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$, $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$. Then the scalar product of these vectors can be found from the formula

$$\mathbf{a} \cdot \mathbf{b} = x_1x_2 + y_1y_2 + z_1z_2.$$

2.3.2. A vector product. The *vector product* of the vector $\mathbf{a}$ by the vector $\mathbf{b}$ is the third vector $\mathbf{c}$ determined as follows (Fig. 18):

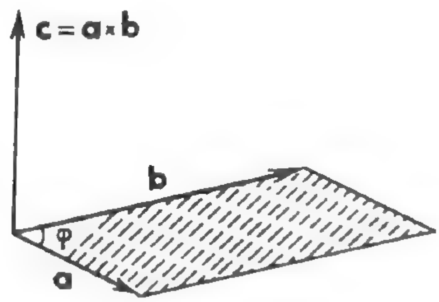

Fig. 18

(1) the absolute value of the vector $\mathbf{c}$ is equal to the area of a parallelogram constructed on the vectors $\mathbf{a}$ and $\mathbf{b}$ ($c = ab \sin\varphi$, where φ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$);
(2) the vector $\mathbf{c}$ is perpendicular to the vectors $\mathbf{a}$ and $\mathbf{b}$;
(3) then the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are translated to a common origin, they become oriented relative to one another as the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ respectively (in the right-handed system of coordinates they form the so-called *right-handed* triad of vectors).

The vector product of $\mathbf{a}$ by $\mathbf{b}$ is designated as $\mathbf{a} \times \mathbf{b}$.

Properties of a Vector Product

1°. $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$, that is, a vector product does not possess a commutative property.

2°. $\mathbf{a} \times \mathbf{b} = 0$ if $\mathbf{a} = 0$, or $\mathbf{b} = 0$, or $\mathbf{a} \parallel \mathbf{b}$.

3°. $(m\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (m\mathbf{b}) = m(\mathbf{a} \times \mathbf{b})$ (an associative property with respect to a scalar multiplier).

4°. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (distributive property).

Vector products of the coordinate unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are:

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0,$$

$$\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k}; \quad \mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j} = \mathbf{i}; \quad \mathbf{k} \times \mathbf{i} = -\mathbf{i} \times \mathbf{k} = \mathbf{j}.$$

The vector product of the vectors $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$ can be most conveniently found from the formula

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}.$$

2.3.3. A mixed product. The *mixed product* of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the scalar product of the vector $\mathbf{a} \times \mathbf{b}$ by the vector $\mathbf{c}$, i.e. $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$.

The mixed product of three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is equal in absolute value to the volume of a parallelepiped constructed on these vectors.

Properties of a Mixed Product

1°. A mixed product of three vectors is equal to zero if:

(a) at least one of the vectors being multiplied is equal to zero;

(b) two of the vectors being multiplied are parallel (collinear);

(c) all three vectors are parallel to one and the same plane (coplanar).

2°. A mixed product remains unchanged if the signs of the vector ($\times$) and scalar ($\cdot$) multiplications in it are interchanged, i.e. $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. By virtue of this property, we shall agree to write the mixed product of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ as $\mathbf{abc}$.

3°. A mixed product does not change if we perform a circular permutation of the vectors:

$$\mathbf{abc} = \mathbf{bca} = \mathbf{cab}.$$

4°. Upon a permutation of any two vectors the mixed product changes only in sign:

$$\mathbf{bac} = -\mathbf{abc}; \quad \mathbf{cba} = -\mathbf{abc}; \quad \mathbf{acb} = -\mathbf{abc}.$$

Suppose the vectors are defined by their resolution with respect to the unit vectors: $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$; $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$; $\mathbf{c} = x_3\mathbf{i} + y_3\mathbf{j} + z_3\mathbf{k}$. Then

$$\mathbf{abc} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

The properties of a mixed product of three vectors imply the following:

the condition $\mathbf{abc} = 0$ is the necessary and sufficient condition of coplanarity of three vectors;

the volume V_1 of the parallelepiped constructed on the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ and the volume V_2 of the triangular pyramid they form can be found by the formula

$$V_1 = |\mathbf{abc}|; \quad V_2 = \frac{1}{6} V_1 = \frac{1}{6} |\mathbf{abc}|.$$

254. Find the scalar product of the vectors $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$.

Solution. We find $\mathbf{a} \cdot \mathbf{b} = 3 \cdot 2 + 4(-5) + 7 \cdot 2 = 0$. Since $\mathbf{a} \cdot \mathbf{b} = 0$, it follows that $\mathbf{a} \perp \mathbf{b}$.

255. Given the vectors $\mathbf{a} = m\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + m\mathbf{j} - 7\mathbf{k}$. At what value of m are these vectors perpendicular?

Solution. We find the scalar product of these vectors: $\mathbf{a} \cdot \mathbf{b} = 4m + 3m - 28$; since $\mathbf{a} \perp \mathbf{b}$, we have $\mathbf{a} \cdot \mathbf{b} = 0$. Hence $7m - 28 = 0$, i.e. $m = 4$.

256. Find $(5\mathbf{a} + 3\mathbf{b}) \cdot (2\mathbf{a} - \mathbf{b})$ if $a = 2$, $b = 3$, $\mathbf{a} \perp \mathbf{b}$.

Solution. We have

$$(5\mathbf{a} + 3\mathbf{b})(2\mathbf{a} - \mathbf{b}) = 10\mathbf{a}^2 - 5\mathbf{a} \cdot \mathbf{b} + 6\mathbf{a} \cdot \mathbf{b} - 3\mathbf{b}^2 = 10a^2 - 3b^2 = 40 - 27 = 13.$$

257. Determine the angle between the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

Solution. Since $\mathbf{a} \cdot \mathbf{b} = ab\cos\varphi$, it follows that $\cos\varphi = \dfrac{\mathbf{a} \cdot \mathbf{b}}{ab}$. We have

$$\mathbf{a} \cdot \mathbf{b} = 1 \cdot 6 + 2 \cdot 4 + 3(-2) = 8, \quad a = \sqrt{1 + 4 + 9} = \sqrt{14},$$

$$b = \sqrt{36 + 16 + 4} = 2\sqrt{14}.$$

Consequently, $\cos\varphi = \dfrac{8}{\sqrt{14} \cdot 2\sqrt{14}} = \dfrac{2}{7}$ and $\varphi = \arccos\dfrac{2}{7}$.

258. Find the unit vector of the same direction as the vector $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

Solution. We find the length of the given vector: $|\mathbf{a}| = \sqrt{1 + 4 + 4} = 3$. Since $\mathbf{a}_0 = \mathbf{a}/|\mathbf{a}|$, we have $\mathbf{a}_0 = (1/3)\mathbf{i} + (2/3)\mathbf{j} + (2/3)\mathbf{k}$.

259. Find the vector product of the vectors $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

Solution. We have

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 5 \\ 1 & 2 & 1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 3 & 5 \\ 2 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix},$$

i.e. $\mathbf{a}\times\mathbf{b} = -7\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

260. Compute the area of a parallelogram constructed on the vectors $\mathbf{a} = 6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$.

Solution. We find the vector product of $\mathbf{a}$ by $\mathbf{b}$:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 3 & -2 \\ 3 & -2 & 6 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 3 & -2 \\ -2 & 6 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 6 & -2 \\ 3 & 6 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 6 & 3 \\ 3 & -2 \end{vmatrix}$$

$$= 14\mathbf{i} - 42\mathbf{j} - 21\mathbf{k}.$$

Since the absolute value of a vector product of two vectors is equal to the area of the parallelogram constructed on them, we have

$$S = |\mathbf{a}\times\mathbf{b}| = \sqrt{14^2 + 42^2 + 21^2} = 49 \text{ (sq. units)}.$$

261. Compute the area of the triangle with the vertices $A(1; 1; 1)$, $B(2; 3; 4)$, $C(4; 3; 2)$.

Solution. We find the vectors $\overline{AB}$ and $\overline{AC}$:

$$\overline{AB} = (2 - 1)\mathbf{i} + (3 - 1)\mathbf{j} + (4 - 1)\mathbf{k} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k},$$

$$\overline{AC} = (4 - 1)\mathbf{i} + (3 - 1)\mathbf{j} + (2 - 1)\mathbf{k} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}.$$

The area of the triangle ABC is equal to half the area of the parallelogram constructed on the vectors $\overline{AB}$ and $\overline{AC}$, therefore, the vector product of these vectors is

$$\overline{AB}\times\overline{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix}$$

$$= -4\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}.$$

Consequently

$$S_{ABC} = \frac{1}{2}\,|\overline{AB}\cdot\overline{AC}| = \frac{1}{2}\sqrt{16 + 64 + 16} = \sqrt{24} \text{ (sq. units)}.$$

262. Compute the area of the parallelogram constructed on the vectors $\mathbf{a} + 3\mathbf{b}$ and $3\mathbf{a} + \mathbf{b}$ if $|\mathbf{a}| = |\mathbf{b}| = 1$, $(\widehat{\mathbf{a}, \mathbf{b}}) = 30°$.

Solution. We have

$$(\mathbf{a} + 3\mathbf{b})\times(3\mathbf{a} + \mathbf{b}) = 3\mathbf{a}\times\mathbf{a} + \mathbf{a}\times\mathbf{b} + 9\mathbf{b}\times\mathbf{a} + 3\mathbf{b}\times\mathbf{b}$$

$$= 3\cdot 0 + \mathbf{a}\times\mathbf{b} - 9\mathbf{a}\times\mathbf{b} + 3\cdot 0 = -8\mathbf{a}\times\mathbf{b}$$

(since $\mathbf{a} \times \mathbf{a} = \mathbf{b} \times \mathbf{b} = 0$, $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$). Thus we have

$$S = 8|\mathbf{a} \times \mathbf{b}| = 8 \cdot 1 \cdot 1 \cdot \sin 30° = 4 \text{ (sq. units)}.$$

263. Find the mixed product of the vectors $\mathbf{a} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\mathbf{c} = \mathbf{i} + \mathbf{j} + 4\mathbf{k}$.

Solution. We have

$$\mathbf{abc} = \begin{vmatrix} 2 & -1 & -1 \\ 1 & 3 & -1 \\ 1 & 1 & 4 \end{vmatrix} = 2\begin{vmatrix} 3 & -1 \\ 1 & 4 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}$$

$$= 26 + 5 + 2 = 33.$$

264. Show that the vectors $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{c} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ are coplanar.

Solution. We find the mixed product of the vectors:

$$\mathbf{abc} = \begin{vmatrix} 2 & 5 & 7 \\ 1 & 1 & -1 \\ 1 & 2 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} - 5\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} + 7\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 8 - 15 + 7 = 0.$$

Since $\mathbf{abc} = 0$, the given vectors are coplanar.

265. Find the volume of the triangular pyramid with the vertices $A(2; 2; 2)$, $B(4; 3; 3)$, $C(4; 5; 4)$ and $D(5; 5; 6)$.

Solution. We find the vectors $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$ coinciding with the edges of the pyramid which converge to the vertex A:

$$\overline{AB} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}, \quad \overline{AC} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}, \quad \overline{AD} = 3\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}.$$

Next we find the mixed product of these vectors:

$$\overline{AB}\,\overline{AC}\,\overline{AD} = \begin{vmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{vmatrix} = 2\begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix}$$

$$= 2 \cdot 6 - 1 \cdot 2 - 1 \cdot 3 = 7.$$

Since the volume of the pyramid is equal to 1/6 of the volume of the parallelepiped constructed on the vectors $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$, we have $V = 7/6$ (cubic units).

266. Calculate $(\mathbf{a} - \mathbf{b})(\mathbf{b} - \mathbf{c})(\mathbf{c} - \mathbf{a})$.

Solution. Since $(\mathbf{a} - \mathbf{b}) + (\mathbf{b} - \mathbf{c}) + (\mathbf{c} - \mathbf{a}) = 0$, these vectors are coplanar (Fig. 19). Consequently, their mixed product is equal to zero, i.e. $(\mathbf{a} - \mathbf{b})(\mathbf{b} - \mathbf{c})(\mathbf{c} - \mathbf{a}) = 0$.

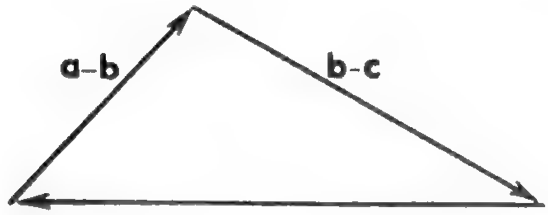

Fig. 19

267. Find the scalar product of the vectors $3\mathbf{a} - 2\mathbf{b}$ and $5\mathbf{a} - 6\mathbf{b}$ if $a = 4$, $b = 6$ and the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ is equal to $\pi/3$.

268. Determine the angle between the vectors $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$.

269. At what value of m are the vectors $\mathbf{a} = m\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ perpendicular?

270. Find the scalar product of the vectors $2\mathbf{a} + 3\mathbf{b} + 4\mathbf{c}$ and $5\mathbf{a} + 6\mathbf{b} + 7\mathbf{c}$ if $a = 1$, $b = 2$, $c = 3$ and $(\widehat{\mathbf{a}, \mathbf{b}}) = (\widehat{\mathbf{a}, \mathbf{c}}) = (\widehat{\mathbf{b}, \mathbf{c}}) = \pi/3$.

271. Find the work done by the force $\mathbf{F}$ along the path $\mathbf{s}$ if $F = 2$, $s = 5$, $\varphi = (\widehat{\mathbf{F}, \mathbf{s}}) = \pi/6$.

272. Find the unit vector perpendicular to the vectors $\mathbf{a} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$.

273. The vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are of the same length and they pairwise form congruent angles. Find the vector $\mathbf{c}$ if $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{j} + \mathbf{k}$.

274. Given the vectors $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$. Find $\mathrm{proj}_{\mathbf{a}}\mathbf{b}$ and $\mathrm{proj}_{\mathbf{b}}\mathbf{a}$.

275. Given the radius vectors of three consecutive vertices of the parallelogram $ABCD$: $\mathbf{r}_A = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{r}_B = \mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, $\mathbf{r}_C = 7\mathbf{i} + 9\mathbf{j} + 11\mathbf{k}$. Determine the radius vector of the fourth vertex D.

276. Show that the vectors $\mathbf{a}$ and $\mathbf{b}$ cannot be perpendicular if $\mathbf{a}\cdot\mathbf{i} > 0$, $\mathbf{a}\cdot\mathbf{j} > 0$, $\mathbf{a}\cdot\mathbf{k} > 0$, $\mathbf{b}\cdot\mathbf{i} < 0$, $\mathbf{b}\cdot\mathbf{j} < 0$, $\mathbf{b}\cdot\mathbf{k} < 0$.

277. Show that the vectors $\mathbf{a} = \mathbf{i} + \mathbf{j} + m\mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + (m + 1)\mathbf{k}$ and $\mathbf{c} = \mathbf{i} - \mathbf{j} + m\mathbf{k}$ cannot be coplanar at any value of m.

278. Can the nonzero numbers $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3$ satisfy the equations

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0, \quad \begin{array}{l} x_1x_2 + y_1y_2 + z_1z_2 = 0, \\ x_1x_3 + y_1y_3 + z_1z_3 = 0, \\ x_2x_3 + y_2y_3 + z_2z_3 = 0? \end{array}$$

279. Find the vector product of the vectors $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

280. Calculate the area of the triangle with the vertices $A(2; 2; 2)$, $B(4; 0; 3)$ and $C(0; 1; 0)$.

281. Find the mixed product of the vectors $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{c} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$.

282. Show that the vectors $\mathbf{a} = 7\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - 7\mathbf{j} + 8\mathbf{k}$, $\mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ are coplanar.

283. Calculate the volume of a triangular pyramid with the vertices $A(0; 0; 1)$, $B(2; 3; 5)$, $C(6; 2; 3)$ and $D(3; 7; 2)$.

284. In the previous problem find the length of the altitude of the pyramid dropped to the face BCD.

285. Show that the points $A(5; 7; -2)$, $B(3; 1; -1)$, $C(9; 4; -4)$ and $D(1; 5; 0)$ lie on the same plane.

Chapter 3

Solid Analytic Geometry

3.1. A Plane and a Straight Line

3.1.1. A plane. (1) *The equation of a plane in vector form* is written as

$$\mathbf{r} \cdot \mathbf{n} = p.$$

Here $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the radius vector of the current point of the plane $M(x; y; z)$; $\mathbf{n} = \mathbf{i} \cos \alpha + \mathbf{j} \cos \beta + \mathbf{k} \cos \gamma$ is a unit vector having the direction of the perpendicular dropped on the plane from the origin; α, β, γ are the angles formed by this perpendicular with the Ox, Oy and Oz axes, and p is the length of the perpendicular.

Upon passing to the coordinates, this equation assumes the form

$$x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0 \tag{1}$$

(*normal form of equation of a plane*).

(2) An equation of any plane can also be written in the form

$$Ax + By + Cz + D = 0. \tag{2}$$

if $A^2 + B^2 + C^2 \neq 0$ (*general form of equation*). Here A, B and C can be regarded as the coordinates of a certain vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ perpendicular to the plane (*normal vector of a plane*). To reduce a general equation of a plane to a normal form, all the terms of the equation must be multiplied by the normalizing factor

$$\mu = \pm 1/N = \pm 1/\sqrt{A^2 + B^2 + C^2}, \tag{3}$$

where the sign before the radical is opposite to the sign of the constant term D in the general equation of a plane.

(3) Special cases of the position of the plane specified by the general equation $Ax + By + Cz + D = 0$:

$A = 0$; parallel to the x-axis;
$B = 0$; parallel to the y-axis;
$C = 0$; parallel to the z-axis;
$D = 0$; passes through the origin;
$A = B = 0$; perpendicular to the z-axis (parallel to the xOy plane);
$A = C = 0$; perpendicular to the y-axis (parallel to the xOz plane);
$B = C = 0$; perpendicular to the x-axis (parallel to the yOz plane);
$A = D = 0$; passes through the x-axis;
$B = D = 0$; passes through the y-axis;
$C = D = 0$; passes through the z-axis;
$A = B = D = 0$; coincides with the xOy plane ($z = 0$);

$A = C = D = 0$; coincides with the xOz plane ($y = 0$);
$B = C = D = 0$; coincides with the yOz plane ($x = 0$).

If in the general equation of a plane the coefficient $D \neq 0$, then, the division of all the terms by $-D$ can reduce the equation to the form

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad (4)$$

(here $a = -D/A$, $b = -D/B$, $c = -D/C$). This is the *intercept form of equation of a plane: a, b* and *c* in it are, respectively, the abscissa, the ordinate and the z-coordinate of the points of intersection of the plane with the Ox, Oy and Oz axes.

(4) *The angle* φ *between the planes* $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is determined by the formula

$$\cos\varphi = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}. \qquad (5)$$

Condition of parallelism of the planes:

$$A_1/A_2 = B_1/B_2 = C_1/C_2. \qquad (6)$$

Condition of perpendicularity of planes:

$$A_1A_2 + B_1B_2 + C_1C_2 = 0. \qquad (7)$$

(5) *The distance from the point* $M_0(x_0;\ y_0;\ z_0)$ *to the plane,* specified by the equation $Ax + By + Cz + D = 0$, can be found from the formula

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}. \qquad (8)$$

It is equal to the absolute value of the result of the substitution of the coordinates of the point into the normal form of equation of the plane; the sign of the result of this substitution shows the mutual position of the point and the origin with respect to the given plane: the plus sign if the point M_0 and the origin lie on the opposite sides of the plane, and the minus sign if they lie on the same side of the plane.

(6) *The equation of a plane which passes through the point* $M_0(x_0;\ y_0;\ z_0)$ *and is perpendicular to the vector* $\mathbf{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0. \qquad (9)$$

With arbitrary values of A, B and C the last equation specifies a certain plane belonging to the pencil of planes passing through the point M_0. It is, therefore, often called the *equation of a pencil of planes.*

(7) The equation

$$A_1x + B_1y + C_1z + D_1 + \lambda(A_2x + B_2y + C_2z + D_2) = 0, \qquad (10)$$

with an arbitrary value of λ, specifies a certain plane passing through the line of in-

tersection of the planes

$$A_1x + B_1y + C_1z + D_1 = 0 \text{ (I)} \quad \text{and } A_2x + B_2y + C_2z + D_2 = 0, \qquad \text{(II)}$$

that is, a certain plane belonging to the pencil of planes passing through that line (therefore, such an equation is often called an *equation of a pencil of planes*). If the planes specified by equations (I) and (II) are parallel, the pencil of planes turns into a collection of planes parallel to these planes.

(8) *Three-point form of equation of a plane passing through the given points* $M_1(\mathbf{r}_1), M_2(\mathbf{r}_2), M_3(\mathbf{r}_3)$, (here $\mathbf{r}_2 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$; $\mathbf{r}_2 = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$; $\mathbf{r}_3 = x_3\mathbf{i} + y_3\mathbf{j} + z_3\mathbf{k}$) can be most conveniently found from the condition of coplanarity of the vectors $\mathbf{r} - \mathbf{r}_1$, $\mathbf{r}_2 - \mathbf{r}_1$, $\mathbf{r}_3 - \mathbf{r}_1$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the radius vector of the current point M of the desired plane:

$$(\mathbf{r} - \mathbf{r}_1)(\mathbf{r}_2 - \mathbf{r}_1)(\mathbf{r}_3 - \mathbf{r}_1) = 0,$$

or in coordinate form

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0. \qquad (11)$$

286. Reduce the equation of a plane $2x + 3y - 6z + 21 = 0$ to the normal form.

Solution. We find the normalizating factor (whose sign is negative since $D = 21 > 0$)

$$\mu = -1/\sqrt{2^2 + 3^2 + 6^2} = -1/7.$$

Thus, the normal form of equation of the given plane is

$$-(2/7)x - (3/7)y + (6/7)z - 3 = 0.$$

287. Determine the distance from the point $M_0(3; 5; -8)$ to the plane $6x - 3y + 2z - 28 = 0$.

Solution. Using formula (8) specifying the distance from a point to a plane, we find

$$d = \frac{|6 \cdot 3 - 3 \cdot 5 + 2 \cdot (-8) - 28|}{\sqrt{6^2 + 3^2 + 2^2}} = \frac{41}{7}.$$

Since the result of the substitution of the coordinates of the point M_0 into the normal form of equation of the plane is negative, the point M_0 and the origin lie on the same side of the given plane.

288. Set up the equation of the plane which passes through the point $M(2; 3; 5)$ and is perpendicular to the vector $\mathbf{N} = 4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

Solution. It suffices to use equation (9) of a plane which passes through a given point and is perpendicular to a given vector:

$$4(x - 2) + 3(y - 3) + 2(z - 5) = 0, \quad \text{i.e.} \quad 4x + 3y + 2z - 27 = 0.$$

289. Find the equation of the plane passing through the point $M(2; 3; -1)$ parallel to the plane $5x - 3y + 2z - 10 = 0$.

Solution. Let us write equation (9) of a pencil of planes passing through a given point:

$$A(x - 2) + B(y - 3) + C(z + 1) = 0.$$

The normal vector of the desired plane coincides with the normal vector $\mathbf{n} = \{5; -3; 2\}$ of the given plane; consequently, $A = 5$, $B = -3$, $C = 2$ and the equation of the sought-for plane has the form

$$5(x - 2) - 3(y - 3) + 2(z + 1) = 0, \quad \text{or} \quad 5x - 3y + 2z + 1 = 0.$$

290. Perpendiculars are dropped from the point $P(2; 3; -5)$ on the axes of coordinates. Set up the equation of the plane passing through their feet.

Solution. The following points serve as the feet of the perpendiculars dropped on the axes of coordinates: $M_1(2; 3; 0)$, $M_2(2; 0; -5)$, $M_3(0; 3; -5)$. Using equation (11), write the equation of the plane passing through the points M_1, M_2, M_3:

$$\begin{vmatrix} x - 2 & y - 3 & z \\ 0 & -3 & -5 \\ -2 & 0 & -5 \end{vmatrix} = 0, \quad \text{or} \quad 15x + 10y - 6z - 60 = 0.$$

291. Set up the equation of the plane passing through the point $A(5; 4; 3)$ and intercepting congruent segments on the coordinate axes.

Solution. We employ the intercept form of equation (4) in which $a = b = c$:

$$\frac{x}{a} + \frac{y}{a} + \frac{z}{a} = 1.$$

The coordinates of the point A satisfy the equation of the desired plane and consequently the equality $5/a + 4/a + 3/a = 1$ is satisfied, whence $a = 12$. Thus we obtain an equation $x + y + z - 12 = 0$.

292. Set up the equation of the plane passing through the line of intersection of the planes $x + y + 5z - 1 = 0$, $2x + 3y - z + 2 = 0$ and through the point $M(3; 2; 1)$.

Solution. Let us make use of equation (10) of a pencil of planes:

$$x + y + 5z - 1 + \lambda(2x + 3y - z + 2) = 0.$$

The value of λ can be determined from the condition that the coordinates of the point M satisfy this equation:

$$3 + 2 + 5 - 1 + \lambda(6 + 6 - 1 + 2) = 9 + 13\lambda = 0,$$

whence $\lambda = -9/13$. Thus, the sought-for equation has the form

$$x + y + 5z - 1 - \frac{9}{13}(2x + 3y - z + 2) = 0, \text{ or } 5x + 14y - 74z + 31 = 0.$$

293. Set up the equation of the plane which passes through the line of intersection of the planes $x + 3y + 5z - 4 = 0$ and $x - y - 2z + 7 = 0$ and is parallel to the y-axis.

Solution. Let us make use of the equation of a pencil of planes:

$$x + 3y + 5z - 4 + \lambda(x - y - 2z + 7) = 0; \ (1 + \lambda)x + (3 - \lambda)y + (5 - 2\lambda)z + (7\lambda - 4) = 0.$$

The sought-for plane being parallel to the axis of ordinates, the coefficient in y must be equal to zero: $3 - \lambda = 0$, i.e. $\lambda = 3$. Substituting the value of λ we have found into the equation of the pencil of planes, we obtain $4x - z + 17 = 0$.

294. Find the equation of the plane passing through the points $A(2; -1; 4)$ and $B(3; 2; -1)$ at right angles to the plane $x + y + 2z - 3 = 0$.

Solution. As the normal vector $\mathbf{N}$ of the desired plane we can take a vector perpendicular to the vector $\overline{AB} = \{1; 3; -5\}$ and to the normal vector $\mathbf{n} = \{1; 1; 2\}$ of the given plane. Therefore, we shall take as $\mathbf{N}$ the vector product of $\overline{AB}$ by $\mathbf{n}$:

$$\mathbf{N} = \overline{AB} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -5 \\ 1 & 1 & 2 \end{vmatrix} = 11\mathbf{i} - 7\mathbf{j} - 2\mathbf{k}.$$

It remains to employ the equation of a plane passing through a given point (say, A) at right angles to the given vector $\mathbf{N} = \{11; -7; -2\}$:

$$11(x - 2) - 7(y + 1) - 2(z - 4) = 0, \quad \text{or} \quad 11x - 7y - 2z - 21 = 0.$$

295. Set up the equation of the plane which passes through the point $M(3; -1; -5)$ and is perpendicular to the planes $3x - 2y + 2z + 7 = 0$ and $5x - 4y + 3z + 1 = 0$.

Solution. It is evident that as the normal vector $\mathbf{N}$ of the desired plane we can take the vector product of the normal vectors $\mathbf{n}_1 = \{3; -2; 2\}$ and $\mathbf{n}_2 = \{5; -4; 3\}$ of the given planes:

$$\mathbf{N} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 2 \\ 5 & -4 & 3 \end{vmatrix} =$$

$$= \mathbf{i}\begin{vmatrix} -2 & 2 \\ -4 & 3 \end{vmatrix} + \mathbf{j}\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & -2 \\ 5 & -4 \end{vmatrix} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}.$$

Using now the equation of the plane passing through the given point $M(3; -1; -5)$ at right angles to the vector $\mathbf{N} = \{2; 1; -2\}$, we obtain

$$2(x - 3) + (y + 1) - 2(z + 5) = 0, \quad \text{or} \quad 2x + y - 2z - 15 = 0.$$

296. Reduce the equations of the following planes to the normal form: (1) $x + y - z - 2 = 0$; (2) $3x + 5y - 4z + 7 = 0$.

297. Find the distance from the point $M_0(1; 3; -2)$ to the plane $2x - 3y - 4z + 12 = 0$. What is the position of the point M_0 with respect to the plane?

298. Find the length of the perpendicular let fall from the point $M_0(2; 3; -5)$ onto the plane $4x - 2y + 5z - 12 = 0$.

299. Find the equation of the plane passing: (1) through the point $M(-2; 3; 4)$ if it intercepts congruent segments on the coordinate axes; (2) through the point $N(2; -1; 4)$ if it intercepts on the z-axis a segment twice as large as on the x- and the y-axis.

300. Find the equation of the plane which passes through the points $P(2; 0; -1)$ and $Q(1; -1; 3)$ and is perpendicular to the plane $3x + 2y - z + 5 = 0$.

301. Find a point M on the plane $2x - 5y + 2z + 5 = 0$ such that the line OM forms congruent angles with the axes of coordinates.

302. Find the equation of the plane if the point $P(4; -3; 12)$ serves as the foot of the perpendicular dropped from the origin onto this plane.

303. Find the equations of the planes passing through the axes of coordinates at right angles to the plane $3x - 4y + 5z - 12 = 0$.

304. Find the equation of the plane whose points are equidistant from the points $P(1; -4; 2)$ and $Q(7; 1; -5)$.

305. Find the equation of the plane passing through the points $P(0; 2; 0)$ and $Q(2; 0; 0)$ and forming an angle of $60°$ with the plane $x = 0$.

306. Calculate the angle between the planes, passing through the point $M(1; -1; -1)$, one of which contains the x-axis and the other the z-axis.

307. Find the equation of the plane passing through the origin and through the points $P(4; -2; 1)$ and $Q(2; 4; -3)$.

308. Find the equation of the plane passing through the point of intersection of the planes $2x + 2y + z - 7 = 0$, $2x - y + 3z - 3 = 0$, $4x + 5y - 2z - 12 = 0$ and through the points $M(0; 3; 0)$ and $N(1; 1; 1)$.

309. Set up the equation of the plane passing through the line of intersection of the planes $x + 5y + 9z - 13 = 0$, $3x - y - 5z + 1 = 0$ and through the point $M(0; 2; 1)$.

310. Set up the equation of the plane passing through the line of intersection of the planes $x + 2y + 3z - 5 = 0$ and $3x - 2y - z + 1 = 0$ and intercepting congruent segments on the Ox and Oz axes.

311. Set up the equation of the plane passing through the line of intersection of the planes $(1 + \sqrt{2})x + 2y + 2z - 4 = 0$ and $x + y + z + 1 = 0$ and forming an angle of $60°$ with the coordinate plane xOy.

312. Set up the equation of the plane which passes through the line of intersection of the planes $2x - y - 12z - 3 = 0$ and $3x + y - 7z - 2 = 0$ and is perpendicular to the plane $x + 2y + 5z - 1 = 0$.

313. Set up the equation of the plane passing through the line of intersection of the planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ and through the origin.

314. Set up the equation of the plane which passes through the point $M(0; 2; 1)$ and is parallel to the vectors $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.

315. What angle is formed by the vector $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and the plane $x + y + 2z - 4 = 0$?

3.1.2. A straight line. (1) A straight line can be specified by the *equations of two planes*

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0, \\ A_2x + B_2y + C_2z + D_2 = 0, \end{cases}$$

intersecting at the given line.

(2) Eliminating in turn x and y from the above-given equations, we obtain equations $x = az + c$, $y = bz + d$. Here the straight line is specified by *two planes projecting it on the planes xOz and yOz*.

(3) *A two-point form of the equation passing through the points* $M_1(x_1; y_1; z_1)$ and $M_2(x_2; y_2; z_2)$ is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}. \tag{1}$$

(4) The so-called *canonical equations*

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \tag{2}$$

specify the straight line which passes through the point $M(x_1; y_1; z_1)$ and is parallel to the vector $\mathbf{s} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$. In particular, these equations can be written in the form

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma},$$

where α, β and γ are angles formed by the straight line and the coordinate axes. The direction cosines of the straight line can be found from the formulas

$$\cos\alpha = \frac{l}{\sqrt{l^2 + m^2 + n^2}}, \quad \cos\beta = \frac{m}{\sqrt{l^2 + m^2 + n^2}}, \quad \cos\gamma = \frac{n}{\sqrt{l^2 + m^2 + n^2}}. \tag{3}$$

(5) Introducing the parameter t, it is easy to pass from canonical equations to *parametric equations* of a straight line:

$$\begin{cases} x = lt + x_1, \\ y = mt + y_1, \\ z = nt + z_1. \end{cases} \tag{4}$$

(6) *The angle between two straight lines* specified by their canonical equations $(x - x_1)/l_1 = (y - y_1)/m_1 = (z - z_1)/n_1$ and $(x - x_2)/l_2 = (y - y_2)/m_2 = (z - z_2)/n_2$ is determined by the formula

$$\cos\varphi = \frac{l_1l_2 + m_1m_2 + n_1n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2}\,\sqrt{l_2^2 + m_2^2 + n_2^2}}; \tag{5}$$

condition of parallelism of two straight lines:

$$l_1/l_2 = m_1/m_2 = n_1/n_2; \tag{6}$$

condition of perpendicularity of two straight lines:

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0. \tag{7}$$

(7) A necessary and sufficient condition for finding two lines, specified by their canonical equations, on one plane (*condition of coplanarity* of two straight lines):

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0. \tag{8}$$

If the quantities l_1, m_1, n_1 are not proportional to the quantities l_2, m_2, n_2, then the indicated relation is the necessary and sufficient condition for intersection of the two lines in space.

(8) *The angle between the line* $(x - x_1)/l = (y - y_1)/m = (z - z_1)/n$ *and the plane* $Ax + By + Cz + D = 0$ is determined by the formula

$$\sin\varphi = \frac{Al + Bm + Cn}{\sqrt{A^2 + B^2 + C^2} \cdot \sqrt{l^2 + m^2 + n^2}}; \tag{9}$$

condition of parallelism of the straight line and the plane:

$$Al + Bm + Cn = 0; \tag{10}$$

condition of perpendicularity of the straight line and the plane:

$$A/l = B/m = C/n. \tag{11}$$

(9) To determine the *point of intersection of the line* $(x - x_0)/l = (y - y_0)/m = (z - z_0)/n$ *and the plane* $Ax + By + Cz + D = 0$, it is necessary to solve their equations simultaneously for which purpose use should be made of the parametric equations of the straight line $x = lt + x_0$, $y = mt + y_0$, $z = nt + z_0$:

(a) if $Al + Bm + Cn \neq 0$, the line intersects the plane;

(b) if $Al + Bm + Cn = 0$ and $Ax_0 + By_0 + Cz_0 + D \neq 0$, the line is parallel to the plane;

(c) if $Al + Bm + Cn = 0$ and $Ax_0 + By_0 + Cz_0 + D = 0$, the line lies on the plane.

316. Reduce the equations of the straight lines $2x - y + 3z - 1 = 0$ and $5x + 4y - z - 7 = 0$ to the canonical form.

Solution. First method. Eliminating first y and then z, we obtain

$$13x + 11z - 11 = 0 \text{ and } 17x + 11y - 22 = 0.$$

If we solve each equation with respect to x, we shall have

$$x = \frac{11(y - 2)}{-17} = \frac{11(z - 1)}{-13}, \quad \text{i.e.} \quad \frac{x}{-11} = \frac{y - 2}{17} = \frac{z - 1}{13}.$$

Second method. We find the vector $\mathbf{s} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$ which is parallel to the sought-for straight line. Since it must be perpendicular to the normal vectors

$\mathbf{N}_1 = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{N}_2 = 5\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ of the given planes, we can take as $\mathbf{s}$ the vector product of the vectors $\mathbf{N}_1$ and $\mathbf{N}_2$:

$$\mathbf{s} = \mathbf{N}_1 \times \mathbf{N}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 5 & 4 & -1 \end{vmatrix} = -11\mathbf{i} + 17\mathbf{j} + 13\mathbf{k}.$$

Thus, $l = -11$, $m = 17$, $n = 13$.

We can take as the point $M_1(x_1; y_1; z_1)$, through which the desired line passes, its point of intersection with any of the coordinate planes, say the yOz plane. Since in this case $x_1 = 0$, we can find the coordinates y_1 and z_1 of that point from the system of equations of the given planes if we put $x = 0$ in them:

$$\begin{cases} -y + 3z - 1 = 0, \\ 4y - z - 7 = 0. \end{cases}$$

Solving this system, we find $y_1 = 2$, $z_1 = 1$. Thus, the desired straight line is determined by the equations $x/(-11) = (y - 2)/17 = (z - 1)/13$.

317. Construct the straight line

$$\begin{cases} 2x + 3y + 3z - 9 = 0, \\ 4x + 2y + z - 8 = 0. \end{cases}$$

Solution. The sought-for straight line can be constructed as the line of intersection of planes. For that purpose we shall write the equations of the planes in the intercept form:

$$\frac{x}{4.5} + \frac{y}{3} + \frac{z}{3} = 1, \quad \frac{x}{2} + \frac{y}{4} + \frac{z}{8} = 1.$$

Having constructed the planes, we obtain the desired straight line (Fig. 20).

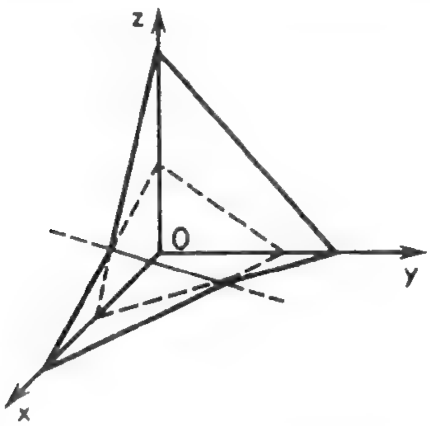

Fig. 20

318. Drop a perpendicular from the origin on the straight line $(x - 2)/2 = (y - 1)/3 = (z - 3)/1$.

Solution. Using condition (11) of perpendicularity of a line and a plane and putting $A = l$, $B = m$, $C = n$, $D = 0$, we set up the equation of the plane which

passes through the origin and is perpendicular to the given straight line. The equation has the form $2x + 3y + z = 0$.

Next we find the point of intersection of the plane and the given line. The parametric equations of the line are $x = 2t + 2$, $y = 3t + 1$, $z = t + 3$. To determine t, we have the equation

$$2(2t + 2) + 3(3t + 1) + t + 3 = 0,$$

whence $t = -5/7$. The coordinates of the point of intersection are $x = 4/7$, $y = -8/7$, $z = 16/7$, i.e. $M(4/7; -8/7; 16/7)$.

It remains to set up the equations of the straight line passing through the origin and through the point M; using relation (1), we obtain

$$x/(4/7) = y/(-8/7) = z/(16/7), \quad \text{or} \quad x/1 = y/(-2) = z/4.$$

319. In the equations of the straight line $x/2 = y/(-3) = z/n$ determine the parameter n so that the given line should intersect the line $(x + 1)/3 = (y + 5)/2 = z/1$ and find the point of their intersection.

Solution. To find the parameter n, we shall use condition (8) for intersection of two straight lines, putting $x_1 = -1$, $y_1 = -5$, $z_1 = 0$, $x_2 = 0$, $y_2 = 0$, $z_2 = 0$, $l_1 = 3$, $m_1 = 2$, $n_1 = 1$, $l_2 = 2$, $m_2 = -3$, $n_2 = n$, we obtain

$$\begin{vmatrix} 1 & 5 & 0 \\ 3 & 2 & 1 \\ 2 & -3 & n \end{vmatrix} = 0, \quad \text{or} \quad 2n + 10 + 3 - 15n = 0, \text{ i.e. } n = 1.$$

To find the coordinates of the point of intersection of the lines $x/2 = y/(-3) = z/1$ and $(x + 1)/3 = (y + 5)/2 = z/1$, we shall express x and y from the first equations in terms of z: $x = 2z$, $y = -3z$. Substituting these values into the equality $(x + 1)/3 = (y + 5)/2$, we obtain $(2z + 1)/3 = (-3z + 5)/2$, whence $z = 1$. Knowing z, we find $x = 2z = 2$, $y = -3z = -3$. Consequently, $M(2; -3; 1)$.

320. Set up the equations of the straight line passing through the point $M(3; 2; -1)$ and intersecting the x-axis at a right angle.

Solution. Since the line is perpendicular to the x-axis and intersects it, it passes through the point $N(3; 0; 0)$. Setting up the equations of the line passing through the points M and N, we obtain $(x - 3)/0 = (y - 2)/(-2) = (z + 1)/1$.

321. Given the plane $x + y - 2z - 6 = 0$ and the point $M(1; 1; 1)$ outside it. Find the point N symmetric with respect to the point M about the given plane.

Solution. Let us write the equations of any line passing through the point M: $(x - 1)/l = (y - 1)/m = (z - 1)/n$. The coordinates $\{l; m; n\}$ of the direction vector of the line perpendicular to the plane can be replaced by the coordinates of the normal vector $\boldsymbol{n} = \{1; 1; -2\}$ of the given plane. Then the equations of that line will be written in the form

$$(x - 1)/1 = (y - 1)/1 = (z - 1)/(-2).$$

We find the projection of the point M on the given plane by solving simultaneous-

ly the equations

$$x + y - 2z - 6 = 0, \quad (x - 1)/1 = (y - 1)/1 = (z - 1)/(-2).$$

Next we rewrite the equations of the line in the form $x = t + 1$, $y = t + 1$, $z = -2t + 1$. Substituting these expressions for x, y and z into the equation of the plane, we obtain $t = 1$, whence $x = 2$, $y = 2$, $z = -1$.

The coordinates of the symmetric point will be found from the equations

$$\bar{x} = (x_M + x_N)/2, \quad \bar{y} = (y_M + y_N)/2, \quad \bar{z} = (z_M + z_N)/2,$$

that is

$$2 = (1 + x_N)/2, \quad 2 = (1 + y_N)/2, \quad -1 = (1 + z_N)/2,$$

whence $x_N = 3$, $y_N = 3$, $z_N = -3$. Consequently, $N(3; 3; -3)$.

322. Given the straight line $(x - 1)/2 = y/3 = (z + 1)/(-1)$ and the point $M(1; 1; 1)$ outside it. Find the point N symmetric with respect to the point M about the given line.

Solution. The equation of the plane which projects the point M on the given line has the form

$$A(x - 1) + B(y - 1) + C(z - 1) = 0.$$

We replace the coordinates $\{A; B; C\}$ of the normal vector of the plane perpendicular to the straight line by the coordinates $\{2; 3; -1\}$ of the direction vector of the given line and obtain

$$2(x - 1) + 3(y - 1) - (z - 1) = 0, \quad \text{or} \quad 2x + 3y - z - 4 = 0.$$

To find the projection of the point M on the straight line, we simultaneously solve the system of equations

$$2x + 3y - z - 4 = 0, \quad (x - 1)/2 = y/3 = (z + 1)/(-1).$$

The parametric equations of the given line are of the form $x = 2t + 1$, $y = 3t$, $z = -t - 1$. Substituting x, y and z into the equation of the plane, we find $t = 1/14$. Hence $x = 8/7$, $y = 3/14$, $z = -15/14$.

Then the coordinates of the symmetric point can be found by using the formulas for the coordinates of the midpoint of the line segment, i.e.

$$8/7 = (1 + x_N)/2, \quad 3/14 = (1 + y_N)/2, \quad -15/14 = (1 + z_N)/2,$$

from which we have $x_N = 9/7$, $y_N = -4/7$, $z_N = -22/7$. Consequently, $N(9/7; -4/7; -22/7)$.

323. Through the line $(x + 1)/2 = (y - 1)/(-1) = (z - 2)/3$ draw a plane parallel to the line $x/(-1) = (y + 2)/2 = (z - 3)/(-3)$.

Solution. Let us write the equation of the first of the given lines using the equations of two planes projecting it on the planes xOy and yOz respectively:

$$(x + 1)/2 = (y - 1)/(-1), \quad \text{or} \quad x + 2y - 1 = 0;$$

$$(y - 1)/(-1) = (z - 2)/3, \quad \text{or} \quad 3y + z - 5 = 0.$$

The equation of the pencil of planes passing through this line has the form

$$x + 2y - 1 + \lambda(3y + z - 5) = 0, \text{ or } x + (2 + 3\lambda)y + \lambda z - (1 + 5\lambda) = 0.$$

Using the condition of parallelism of a line and a plane, let us define λ so that the corresponding plane of the pencil should be parallel to the second of the given lines. We have $-1\cdot 1 + 2(2 + 3\lambda) - 3\lambda = 0$, or $3\lambda + 3 = 0$, whence $\lambda = -1$. Thus, the desired plane is specified by the equation $x - y - z + 4 = 0$.

324. Find the equations of the projection of the straight line $(x - 1)/1 = (y + 1)/2 = z/3$ on the plane $x + y + 2z - 5 = 0$.

Solution. We write the equations of the given line as the equations of two planes projecting it on the planes xOy and xOz respectively:

$$(x - 1)/1 = (y + 1)/2, \quad \text{or} \quad 2x - y - 3 = 0;$$
$$(x - 1)/1 = z/3, \quad \text{or} \quad 3x - z - 3 = 0.$$

The equation of the pencil of planes passing through the given line will be written in the form

$$2x - y - 3 + \lambda(3x - z - 3) = 0, \quad \text{or} \quad (2 + 3\lambda)x - y - \lambda z - 3(1 + \lambda) = 0.$$

Using the condition of perpendicularity of planes, we choose a plane from the pencil which projects the given line on the given plane. We obtain $1\cdot(2 + 3\lambda) + 1(-1) + 2(-\lambda) = 0$, or $\lambda + 1 = 0$, whence $\lambda = -1$. Thus, the equation of the projecting plane has the form

$$2x - y - 3 + (-1)\cdot(3x - z - 3) = 0, \quad \text{or} \quad x + y - z = 0.$$

The sought-for projection can be defined as the line of intersection of two planes, the given plane and the projecting one:

$$\begin{cases} x + y + 2z - 5 = 0, \\ x + y - z = 0. \end{cases}$$

Reducing these equations of a line to the canonical form, we finally obtain

$$x/1 = (y - 5/3)/(-1) = (z - 5/3)/0.$$

325. Set up the equations of the straight line which passes through the point $M(5; 3; 4)$ and is parallel to the vector $\mathbf{s} = 2\mathbf{i} + 5\mathbf{j} - 8\mathbf{k}$.

Solution. Let us make use of the canonical equations of a straight line. Putting $l = 2$, $m = 5$, $n = 8$, $x_1 = 5$, $y_1 = 3$, $z_1 = 4$ in equalities (2), we obtain

$$(x - 5)/2 = (y - 3)/5 = (z - 4)/(-8).$$

326. Set up the equations of the line which passes through the point $M(1; 1; 1)$ and is perpendicular to the vectors $\mathbf{s}_1 = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{s}_2 = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

Solution. The line is parallel to the vector $\mathbf{s}_1 \times \mathbf{s}_2 = 5\mathbf{i} - \mathbf{j} - 7\mathbf{k}$, therefore it is specified by the equations

$$(x - 1)/5 = (y - 1)/(-1) = (z - 1)/(-7).$$

327. Find the equations of the projections of the line

$$\begin{cases} x + 2y + 3z - 26 = 0, \\ 3x + y + 4z - 14 = 0 \end{cases}$$

on the coordinate planes.

328. Reduce to the canonical form the equations of the straight line

$$\begin{cases} 2x + 3y - 16z - 7 = 0, \\ 3x + y - 17z = 0. \end{cases}$$

329. Calculate the angles formed by the line

$$\begin{cases} x - 2y - 5 = 0, \\ x - 3z + 8 = 0 \end{cases}$$

with the axes of coordinates.

330. Find the equations of the straight line passing through the point $M(1; -2; 3)$ and forming angles of 45° and 60° with the Ox and Oy axes.

331. Find the equations of the straight line which passes through the point $N(5; -1; -3)$ and is parallel to the line

$$\begin{cases} 2x + 3y + z - 6 = 0, \\ 4x - 5y - z + 2 = 0. \end{cases}$$

332. Find the point of intersection of the straight lines $(x - 1)/(-1) = (y - 2)/5 = (z + 4)/2$ and $(x - 2)/2 = (y - 5)/(-2) = (z - 1)/3$.

333. Given three consecutive vertices of a parallelogram: $A(3; 0; -1)$, $B(1; 2; -4)$ and $C(0; 7; -2)$. Find the equations of the sides AD and CD.

334. Find the parametric equations of the line passing through the points $M(2; -5; 1)$ and $N(-1; 1; 2)$.

335. Calculate the distance between the parallel lines $x/1 = (y - 3)/2 = (z - 2)/1$ and $(x - 3)/1 = (y + 1)/2 = (z - 2)/1$.

336. Given the points $A(-1; 2; 3)$ and $B(2; -3; 1)$. Set up the equations of the line which passes through the point $M(3; -1; 2)$ and is parallel to the vector $\overline{AB}$.

337. Find the angle between the straight lines

$$\begin{cases} 4x - y - z + 12 = 0, \\ y - z - 2 = 0 \end{cases} \quad \text{and} \quad \begin{cases} 3x - 2y + 16 = 0, \\ 3x - z = 0. \end{cases}$$

338. Find the line, in the yOz plane, which passes through the origin and is perpendicular to the line $\begin{cases} 2x - y = 2, \\ y + 2z = -2. \end{cases}$

339. Given two vertices of the parallelogram $ABCD$: $C(-2; 3; -5)$ and $D(0; 4; -7)$ and the point of intersection of the diagonals $M(1; 2; -3. 5)$. Find the equations of the side AB.

340. The triangle ABC is formed as a result of intersection of the plane $x + 2y + 4z - 8 = 0$ and the coordinate axes. Find the equations of the median of the triangle which is parallel to the xOy plane.

341. Given the points $A(1; 1; 1)$, $B(2; 3; 3)$ and $C(3; 3; 2)$. Set up the equations of the line which passes through the point A and is perpendicular to the vectors $\overline{AB}$ and $\overline{AC}$.

342. Set up the equations of the straight line passing through the point $M(0; 2; 1)$ and forming congruent angles with the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 3\mathbf{j}$, $\mathbf{c} = 3\mathbf{k}$.

343. Find the equation of the plane which passes through the straight line $(x + 1)/3 = (y - 2)/(-1) = z/4$ and is perpendicular to the plane $3x + y - z + 2 = 0$.

344. Find the equations of the projection of the straight line $x/2 = (y + 3)/1 = (z - 2)/(-2)$ on the plane $2x + 3y - z - 5 = 0$.

3.2. Surfaces of the Second Order

3.2.1. A sphere. In the Cartesian system of coordinates, a sphere with centre at the point $C(a; b; c)$ and radius r is specified by the equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2. \tag{1}$$

If the centre of the sphere is at the origin, its equation has the form

$$x^2 + y^2 + z^2 = r^2. \tag{2}$$

345. Find the coordinates of the centre and the radius of the sphere specified by the equation $x^2 + y^2 + z^2 - x + 2y + 1 = 0$.

Solution. Reduce the equation of the sphere to the canonical form (1) for which purpose complete the terms containing x, y and z to perfect squares, that is, rewrite the equation in the form

$$\left(x^2 - x + \frac{1}{4}\right) - \frac{1}{4} + (y^2 + 2y + 1) - 1 + z^2 + 1 = 0,$$

or

$$\left(x - \frac{1}{2}\right)^2 + (y + 1)^2 + z^2 = \frac{1}{4}.$$

Consequently, the centre of the sphere is the point $C(1/2; -1; 0)$ and its radius is $r = 1/2$.

346. Set up the equation of the sphere passing through the points $A(1; 2; -4)$, $B(1; -3; 1)$ and $C(2; 2; 3)$ if its centre is on the plane xOy.

Solution. Since the points A, B and C belong to the sphere $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$ whose centre is on the plane xOy (whence $c = 0$), their coordinates must turn the sought-for equation into identity; therefore, we obtain the following equations:

$$(1 - a)^2 + (2 - b)^2 + (-4)^2 = r^2,\ (1 - a)^2 + (-3 - b)^2 + 1^2 = r^2,$$
$$(2 - a)^2 + (2 - b)^2 + 3^2 = r^2.$$

From this we have

$$(1 - a)^2 + (2 - b)^2 + 16 = (1 - a)^2 + (-3 - b)^2 + 1,$$
$$(1 - a)^2 + (2 - b)^2 + 16 = (2 - a)^2 + (2 - b)^2 + 9,$$

or

$$(2 - b)^2 - (-3 - b)^2 = -15, \text{ i.e. } 10b = 10;$$
$$(1 - a)^2 - (2 - a)^2 = -7, \text{ i.e. } 2a = -4.$$

Thus we have $a = -2$, $b = 1$. Consequently, the centre of the sphere is the point $C(-2;\ 1;\ 0)$. Furthermore, we find $r^2 = (1 - a)^2 + (2 - b)^2 + 16 = (1 + 2)^2 + (2 - 1)^2 + 16 = 26$. Thus, the desired equation has the form

$$(x + 2)^2 + (y - 1)^2 + z^2 = 26.$$

347. Find the coordinates of the centre and the radius of the circle

$$\begin{cases} (x - 3)^2 + (y + 2)^2 + (z - 1)^2 = 100, \\ 2x - 2y - z + 9 = 0. \end{cases}$$

Solution. Let us drop a perpendicular from the centre of the sphere $C(3;\ -2;\ 1)$ on the plane $2x - 2y - z + 9 = 0$, the equations of the perpendicular being

$$(x - 3)/2 = (y + 2)/(-2) = (z - 1)/(-1) \qquad (*)$$

(the normal vector of the given plane can be taken as the direction vector of the perpendicular).

Next we find the coordinates of the point of intersection of the line $(*)$ and the plane $2x - 2y - z + 9 = 0$. This point is precisely the centre of the circle which is the intersection of the sphere and the given plane.

Having written the equations of the line in the parametric form $x = 2t + 3$, $y = -2t - 2$, $z = -t + 1$, we can substitute x, y and z into the equation of the plane and obtain

$$2(2t + 3) - 2(-2t - 2) - (-t + 1) + 9 = 0, \quad \text{i.e.} \quad t = -2.$$

Consequently, $x = 2(-2) + 3 = -1$, $y = -2(-2) - 2 = 2$, $z = -(-2) + 1 = 3$, that is, the centre of the sphere is at the point $C_1(-1;\ 2;\ 3)$.

Let us now find the distance d from the centre of the sphere $C(3;\ -2;\ 1)$ to the plane $2x - 2y - z + 9 = 0$:

$$d = \frac{2\cdot 3 + 2\cdot 2 - 1 + 9}{\sqrt{2^2 + 2^2 + 1}} = 6.$$

The radius r of the circle can be determined from the equality $r^2 = R^2 - d^2$, where R is the radius of the sphere; thus we have $r^2 = 100 - 36 = 64$, i.e. $r = 8$.

348. Determine the coordinates of the centres and the radii of the spheres specified by the equations (1)$(x + 1)^2 + (y + 2)^2 + z^2 = 25$; (2) $x^2 + y^2 + z^2 - 4x + 6y + 2z - 2 = 0$; (3) $2x^2 + 2y^2 + 2z^2 + 4y - 3z + 2 = 0$; (4) $x^2 + y^2 + z^2 = 2x$; (5) $x^2 + y^2 + z^2 = 4z - 3$.

349. What is the position of the point $M(1;\ -1;\ 3)$ with respect to the spheres (1) $(x - 1)^2 + (y + 2)^2 + z^2 = 19$; (2) $x^2 + y^2 + z^2 - x + y = 0$; (3) $x^2 + y^2 + z^2 - 4x + y - 2z = 0$?

350. Set up the equation of the sphere if the points $M(4;\ -1;\ -3)$ and $N(0;\ 3;\ -1)$ are the end points of one of its diameters.

351. Set up the equation of the circle which is formed as a result of the intersection of the sphere $(x - 1)^2 + (y - 1)^2 + (z - 3)^2 = 25$ and the coordinate plane $z = 0$.

352. Find the coordinates of the centre and the radius of the circle $x^2 + y^2 + z^2 = 100$, $2x + 2y - z = 18$.

3.2.2. Cylindrical surfaces and a second-order cone. The equation of the form $F(x, y) = 0$ in space specifies a cylindrical surface whose generating elements are parallel to the z-axis. Similarly, the equation $F(x, z) = 0$ specifies a cylindrical surface with generating elements parallel to the y-axis, and the equation $F(y, z) = 0$, a cylindrical surface whose generating elements are parallel to the x-axis.

The canonical equations of second-order cylinders are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \textit{ elliptic cylinder,}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \textit{ hyperbolic cylinder,}$$

$$y^2 = 2px, \textit{ parabolic cylinder.}$$

The generating elements of the three cylinders specified by these equations are parallel to the z-axis and the directing curve is the corresponding quadratic curve (ellipse, hyperbola, parabola) lying on the xOy plane.

It should be remembered that a curve in space can be defined either parametrically or as a line of intersection of two surfaces. For instance, the equations of the directing curve of an elliptic cylinder, that is, the equations of an ellipse on the xOy plane, are of the form

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, \\ z = 0. \end{cases}$$

The equation of a *second-order cone* with the vertex at the origin whose axis is the z-axis is written in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

Similarly, the equations

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0, \quad -\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$$

are the equations of second-order cones with the vertex at the origin, whose axes are Oy and Ox respectively.

353. What surface is specified in space by the equations (1) $x^2 = 4y$; (2) $z^2 = xz$?

Solution. (1) The equation $x^2 = 4y$ specifies a parabolic cylinder with the elements parallel to the z-axis. The directing curve of the cylindrical surface is the parabola $x^2 = 4y$, $z = 0$.

(2) The equation $z^2 = xz$ can be represented in the form $z(z - x) = 0$ and disintegrates into two equations $z = 0$ and $z = x$, that is, it specifies two planes, the xOy plane and the bisector plane $z = x$ passing through the y-axis.

354. Along what line does the cone $x^2 + y^2 - 2z^2 = 0$ intersect the plane $y = 2$?

Solution. Eliminating y from the system of equations, we obtain

$$x^2 + 4 - 2z^2 = 0, \quad \text{or} \quad z^2/2 - x^2/4 = 1.$$

Consequently, the sought-for line of intersection is a hyperbola lying on the plane $y = 2$; its real axis is parallel to the z-axis and the imaginary axis, to the x-axis.

355. Set up the equation of the conic surface whose vertex is the point $M(0; 0; 1)$ and whose directing curve is the ellipse $x^2/25 + y^2/9 = 1$, $z = 3$.

Solution. We shall set up the equation of the generating element $\dot{A}M$, where $A(x_0; y_0; z_0)$ is a point lying on the ellipse. The equation of this element is of the form $x/x_0 = y/y_0 = (z - 1)/(z_0 - 1)$. Since the point A belongs to the ellipse, its coordinates satisfy the equations of an ellipse, that is, $x_0^2/25 + y_0^2/9 = 1$, $z_0 = 3$.

Eliminating now x_0, y_0 and z_0 form the system

$$x/x_0 = (z - 1)/(z_0 - 1), y/y_0 = (z - 1)/(z_0 - 1), x_0^2/25 + y_0^2/9 = 1, z_0 = 3,$$

we obtain the equation of the desired cone:

$$x^2/25 + y^2/9 - (z - 1)^2/4 = 0.$$

356. Determine what surfaces are specified by the following equations and construct the surfaces: (1) $x^2 + y^2 = 4$; (2) $x^2/25 + y^2/16 = 1$; (3) $x^2 - y^2 = 1$; (4) $y^2 = 2x$; (5) $z^2 = y$; (6) $z + x^2 = 0$; (7) $x^2 + y^2 = 2y$; (8) $x^2 + y^2 = 0$; (9) $x^2 - z^2 = 0$; (10) $y^2 = xy$.

357. Set up the equations of the lines of intersection of the cone $x^2 - y^2 + z^2 = 0$ and the planes (1) $y = 3$; (2) $z = 1$; (3) $x = 0$.

358. Set up the equation of the cone with the vertex at the origin whose directing curves are given by the equations (1) $x = a$, $y^2 + z^2 = b^2$; (2) $y = b$, $x^2 + z^2 = a^2$; (3) $z = c$, $x^2/a^2 + y^2/b^2 = 1$.

3.2.3. Surfaces of revolution. Second-order surfaces. If the curve $F(y, z) = 0$, $x = 0$ lying on the plane yOz rotates about the z-axis, then the equation of the surface of revolution it generates has the form

$$F(\sqrt{x^2 + y^2}, z) = 0.$$

Similarly, the equation $F(x, \sqrt{y^2 + z^2}) = 0$ specifies a surface generated by the rotation of the curve $F(x, y) = 0$, $z = 0$ about the x-axis; the equation $F(\sqrt{x^2 + z^2}, y) = 0$ specifies a surface generated by the rotation of the same curve about the y-axis.

Next follow the examples of the second-order surfaces of revolution generated by the rotation of an ellipse, a hyperbola and a parabola about their axes of symmetry.

An *ellipsoid of revolution* (*spheroid*)

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1;$$

the axis of revolution is the z-axis; a spheroid is oblate at $a > c$ and prolate at $a < c$ (at $a = c$ it turns into a sphere).

Hyperboloid of revolution of one sheet (*unparted hyperboloid*)

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = 1;$$

the axis of revolution is the z-axis (serving as a conjugate axis of the hyperbola whose rotation generates the surface).

Hyperboloid of revolution of two sheets

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = -1;$$

the axis of rotation is the z-axis (serving as the transverse axis of the hyperbola whose rotation generates the surface).

Paraboloid of revolution

$$x^2 + y^2 = 2pz;$$

the axis of revolution is the z-axis.

Surfaces of revolution of the second order are a special case of second-order surfaces of the general form, whose canonical equations are the following:

Ellipsoid (triaxial)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Hyperboloid of two sheets

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1.$$

Elliptic paraboloid

$$\frac{x^2}{p} + \frac{y^2}{q} = 2z \ (p > 0, q > 0).$$

Besides these four second-order surfaces, three second-order cylindres (elliptic, hyperbolic and parabolic) and a second-order cone, there is one more second-order surface, a *hyperbolic paraboloid*, whose canonical equation has the form

$$\frac{x^2}{p} - \frac{y^2}{q} = 2z \ (p > 0, q > 0).$$

Thus there are nine various surfaces of the second order all told.

359. Find the equation of the surface generated by the revolution of the straight line $x + 2y = 4$, $z = 0$ about the x-axis.

Solution. The surface of revolution is a cone with vertex at the point $M(4; 0; 0)$. Suppose an arbitrary point A of the desired surface has the coordinates X, Y, Z; it is associated with a point $B(x; y; 0)$ lying on the given line. The points A and B belong to the same plane which is perpendicular to the axis of rotation Ox. Then we have $X = x$, $Y^2 + Z^2 = y^2$.

Substituting the expressions for x and y into the equation of the given line, we obtain the equation of the desired surface of revolution:

$$X + 2\sqrt{Y^2 + Z^2} = 4, \text{ or } 4(Y^2 + Z^2) - (X - 4)^2 = 0, \quad \text{or}$$
$$4Y^2 + 4Z^2 - (X - 4)^2 = 0.$$

360. What surface is specified by the equation $x^2 = yz$?

Solution. Let us rotate the coordinate axes about the x-axis through an angle $\alpha = 45°$ (from the y-axis to the z-axis counterclockwise). The formulas for the transformation of coordinates are the following: $x = x'$, $y = y'\cos\alpha - z'\sin\alpha$, $z = y'\sin\alpha + z'\cos\alpha$. Since $\sin\alpha = \cos\alpha = \sqrt{2}/2$, we have $x = x'$, $y = (\sqrt{2}/2)(y' - z')$, $z = (\sqrt{2}/2)(y' + z')$. Substituting these expressions into the equation of the surface, we obtain

$$x'^2 = y'^2/2 - z'^2/2, \quad \text{or} \quad x'^2 - y'^2/2 + z'^2/2 = 0$$

(a cone with the vertex at the origin, whose axis is the axis of ordinates).

361. Find the equation of the surface resulting from the rotation of the line $2y + z - 2 = 0$, $x = 0$ about the z-axis.

362. Find the equation of the intersection lines of the surface $z = x^2 - y^2$ and the planes $z = 1$, $y = 1$, $x = 1$, $z = -1$.

363. What surfaces are specified by the equations (1) $z = xy$, (2) $z^2 = xy$?

Hint. Perform a rotation about the z-axis through an angle of 45°.

364. Find the equation of the elliptic paraboloid with its vertex at the origin whose axis is the z-axis if two points, $M(-1; -2; 2)$ and $N(1; 1; 1)$, are defined on its surface.

365. Set up the equation of the ellipsoid whose symmetry axes are the axes of coordinates if three points, $A(3; 0; 0)$, $B(-2; 5/3; 0)$ and $C(0; -1; 2/\sqrt{5})$, are defined on its surface.

366. Find the equations of the line of intersection of the surfaces $z = 2 - x^2 - y^2$ and $z = x^2 + y^2$.

367. Investigate what surfaces are specified by the equation $z^2 + x^2 = m(z^2 + y^2)$ at (1) $m = 0$; (2) $0 < m < 1$; (3) $m > 1$; (4) $m < 0$; (5) $m = 1$.

3.2.4. The general form of equation of a second-order surface. The second-degree general equation with respect to x, y and z has the form

$$Ax^2 + By^2 + Cz^2 + 2Dyz + 2Exz + 2Fxy + 2Gx + 2Hy + 2Kz + L = 0.$$

This equation can specify a sphere, an ellipsoid, a hyperboloid of one sheet or two sheets, an elliptic or hyperbolic paraboloid, a second-order cylindrical or conical surface. It can also specify a set of two planes, a point, a straight line or even be geometrically meaningless (specify an "imaginary" surface).

For $D = 0$, $E = 0$, $F = 0$ the general equation assumes the form

$$Ax^2 + By^2 + Cz^2 + 2Gx + 2Hy + 2Kz + L = 0.$$

In this case the equation can be easily simplified by means of a parallel translation of the coordinate axes which makes it possible to establish at once its geometrical meaning.

368. What is the geometrical meaning of the equation

$$x^2 + 4y^2 + 9z^2 + 12yz + 6xz + 4xy - 4x - 8y - 12z + 3 = 0?$$

Solution. This equation can be written in the form

$$(x + 2y + 3z)^2 - 4(x + 2y + 3z) + 3 = 0.$$

Let us factorize the left-hand side of the equation:

$$(x + 2y + 3z - 1)(x + 2y + 3z - 3) = 0.$$

Thus we see that the equation specifies a set of two planes

$$x + 2y + 3z - 1 = 0 \text{ and } x + 2y + 3z - 3 = 0.$$

369. What is the geometrical meaning of the equation

$$x^2 + y^2 + z^2 - yz - xz - xy = 0?$$

Solution. Multiplying by two, we shall rewrite the equation in the form

$$2x^2 + 2y^2 + 2z^2 - 2yz - 2xz - 2xy = 0,$$

or

$$(x - y)^2 + (y - z)^2 + (x - z)^2 = 0.$$

This equation is satisfied by the coordinates of only those points for which the equalities $x = y$, $y = z$, $x = z$ are satisfied. Thus we see that equation specifies the line $x = y = z$.

370. What is the geometrical meaning of the equation

$$x^2 + y^2 + 4z^2 - 2xy - 8z + 5 = 0?$$

Solution. Let us rewrite the equation in the form

$$(x - y)^2 + 4(z - 1)^2 = -1.$$

This equation is geometrically meaningless since its left-hand side cannot be a negative quantity for any real values of x, y and z.

371. Reduce to the canonical form the equation

$$4x^2 + 9y^2 + 36z^2 - 8x - 18y - 72z + 13 = 0.$$

Solution. Let us collect the terms with the same coordinates:

$$4(x^2 - 2x) + 9(y^2 - 2y) + 36(z^2 - 2z) = -13.$$

Completing the expressions in parentheses to perfect squares, we get

$$4(x^2 - 2x + 1) + 9(y^2 - 2y + 1) + 36(z^2 - 2z + 1) = -13 + 4 + 9 + 36,$$

or

$$4(x - 1)^2 + 9(y - 1)^2 + 36(z - 1)^2 = 36.$$

Next we perform a parallel translation of the coordinate axes taking the point $O'(1; 1; 1)$ as the new origin. The formulas for the transformation of coordinates are of the form $x = x' + 1, y = y' + 1, z = z' + 1$. In that case the equation of the surface can be written as

$$4x'^2 + 9y'^2 + 36z'^2 = 36, \quad \text{or} \quad x'^2/9 + y'^2/4 + z'^2 = 1.$$

This equation specifies an ellipsoid; its centre is at the new origin and the semi-axes are equal, respectively, to 3, 2 and 1.

372. Reduce the following equation to the canonical form:

$$x^2 - y^2 - 4x + 8y - 2z = 0.$$

Solution. Collecting the terms containing x and y, we get

$$(x^2 - 4x) - (y^2 - 8y) = 2z.$$

Next we complete the expressions in parentheses to perfect squares:

$$(x^2 - 4x + 4) - (y^2 - 8y + 16) = 2z + 4 - 16, \text{ or } (x - 2)^2 - (y - 4)^2 = 2(z - 6).$$

Let us perform a parallel translation of the coordinate axes taking the point $O'(2; 4; 6)$ as the new origin. Then we have $x = x' + 2, y = y' + 4, z = z' + 6$. As a result we obtain an equation $x'^2 - y'^2 = 2z'$ specifying a hyperbolic paraboloid.

373. What surface is specified by the equation

$$4x^2 - y^2 + 4z^2 - 8x + 4y + 8z + 4 = 0?$$

Solution. Having carried out the requisite transformations, we get

$$4(x^2 - 2x) - (y^2 - 4y) + 4(z^2 + 2z) = -4;$$

$$4(x^2 - 2x + 1) - (y^2 - 4y + 4) + 4(z^2 + 2z + 1) = -4 + 4 - 4 + 4;$$

$$4(x - 1)^2 - (y - 2)^2 + 4(z + 1)^2 = 0.$$

Next we perform a parallel translation of the coordinate axes taking the point $O'(1; 2; -1)$ as the new origin. The formulas for the transformation of coordinates are $x = x' + 1, y = y' + 2, z = z' - 1$. Then the given equation assumes the form

$$4x'^2 - y'^2 + 4z'^2 = 0, \quad \text{or} \quad x'^2 - y'^2/4 + z'^2 = 0.$$

This is an equation of a conical surface.

Investigate what surfaces are specified by the following equations:

374. $x^2 - xy - xz + yz = 0.$
375. $x^2 + z^2 - 4x - 4z + 4 = 0.$
376. $x^2 + 2y^2 + z^2 - 2xy - 2yz = 0.$
377. $x^2 + y^2 - z^2 - 2y + 2z = 0.$
378. $x^2 + 2y^2 + 2z^2 - 4y + 4z + 4 = 0.$
379. $4x^2 + y^2 - z^2 - 24x - 4y + 2z + 35 = 0.$
380. $x^2 + y^2 - z^2 - 2x - 2y + 2z + 2 = 0.$
381. $x^2 + y^2 - 6x + 6y - 4z + 18 = 0.$
382. $9x^2 - z^2 - 18x - 18y - 6z = 0.$

Chapter 4
Determinants and Matrices

4.1. The Concept of the nth-Order Determinant

The fourth-order determinant corresponding to the table of the elements

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

is specified by the equality

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} -$$

$$- a_{12} \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14} \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}.$$

Fourth-order determinants can be used to introduce by analogy the concept of the fifth-order determinant, and so on.

The definitions of a minor and of a signed minor or cofactor of a certain element as well as both theorems concerning cofactors formulated for third-order determinants hold true for determinants of any order.

Thus, denoting by M_{jk} the minor and by A_{jk} the cofactor of the element a_{jk} of the nth-order determinant (that is, of the element located in the jth row and the kth column of that determinant), we get

$$A_{jk} = (-1)^{j+k} M_{jk}.$$

Assume that D is the nth-order determinant. Expanding it first by the elements of the jth row and then by the elements of the kth column, we get, by virtue of the first theorem on cofactors,

$$D = a_{j1}A_{j1} + a_{j2}A_{j2} + \dots + a_{jn}A_{jn};$$

$$D = a_{1k}A_{1k} + a_{2k}A_{2k} + \dots + a_{nk}A_{nk}.$$

On the other hand, by virtue of the second theorem on cofactors we have at $j \neq i$ and $k \neq l$

$$a_{j1}A_{i1} + a_{j2}A_{i2} + \dots + a_{jn}A_{in} = 0,$$
$$a_{1k}A_{1l} + a_{2k}A_{2l} + \dots + a_{nk}A_{nl} = 0.$$

The properties of determinants of the second and the third order formulated in 1.5.2 also hold true for determinants of any order.

The solution of the system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n, \end{cases}$$

whose determinant

$$D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \neq 0,$$

can be found by the formulas

$$x_1 = \frac{D_1}{D}; \quad x_2 = \frac{D_2}{D}; \quad \dots; \quad x_n = \frac{D_n}{D}.$$

In these formulas, D is the determinant of the system and D_k ($k = 1, 2, \dots, n$) is the determinant which can be obtained from the determinant of the system by replacing the kth column (that is, the column of coefficients in the unknown being sought) by the column of constant terms:

$$D_k = \begin{vmatrix} a_{11} & a_{12} \dots a_{1,\,k-1} b_1 & a_{1,\,k+1} \dots a_{1n} \\ a_{21} & a_{22} \dots a_{2,\,k-1} b_2 & a_{2,\,k+1} \dots a_{2n} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ a_{n1} & a_{n2} \dots a_{n,\,k-1} b_n & a_{n,\,k+1} \quad a_{nn} \end{vmatrix}.$$

383. Calculate the determinant

$$\begin{vmatrix} 3 & 5 & 7 & 2 \\ 1 & 2 & 3 & 4 \\ -2 & -3 & 3 & 2 \\ 1 & 3 & 5 & 4 \end{vmatrix}.$$

Solution. Perform the following actions: (1) subtract the trebled elements of the 2nd row from the elements of the 1st row (2) add the doubled elements of the 2nd row to the elements of the 3rd row (3) subtract the elements of the 2nd row from the elements of the 4th row. Then the original determinant will reduce to the following form:

$$D = \begin{vmatrix} 0 & -1 & -2 & -10 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 9 & 10 \\ 0 & 1 & 2 & 0 \end{vmatrix}.$$

Next expand the determinant by the elements of the 1st column:

$$D = -\begin{vmatrix} -1 & -2 & -10 \\ 1 & 9 & 10 \\ 1 & 2 & 0 \end{vmatrix}.$$

Adding the elements of the 3rd row to those of the 1st row and subtracting the elements of the 3rd row from those of the 2nd row, we get

$$D = -\begin{vmatrix} 0 & 0 & 10 \\ 0 & 7 & 10 \\ 1 & 2 & 0 \end{vmatrix}.$$

Now expand the determinant by the elements of the 1st column:

$$D = -\begin{vmatrix} 0 & 10 \\ 7 & 10 \end{vmatrix} = 70.$$

384. Calculate the determinant

$$\begin{vmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 2 & 3 & 0 & 0 \\ 0 & 4 & 3 & 4 & 0 \\ 0 & 0 & 5 & 4 & 5 \\ 0 & 0 & 0 & 6 & 5 \end{vmatrix}.$$

Solution. Place the common multipliers of the 2nd, 4th and 5th columns outside the determinant

$$D = 2 \cdot 2 \cdot 5 \cdot \begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ 3 & 1 & 3 & 0 & 0 \\ 0 & 2 & 3 & 2 & 0 \\ 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 3 & 1 \end{vmatrix}$$

and subtract the elements of the 1st column from the elements of the 2nd column:

$$D = 20 \cdot \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & -2 & 3 & 0 & 0 \\ 0 & 2 & 3 & 2 & 0 \\ 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 3 & 1 \end{vmatrix}.$$

Next expand the determinant by the elements of the 1st row:

$$D = 20 \cdot \begin{vmatrix} -2 & 3 & 0 & 0 \\ 2 & 3 & 2 & 0 \\ 0 & 5 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{vmatrix}.$$

Now add the elements of the 1st row to those of the 2nd row and place -2 (the common multiplier of the elements of the 1st column) outside the determinant:

$$D = -40 \cdot \begin{vmatrix} 1 & 3 & 0 & 0 \\ 0 & 6 & 2 & 0 \\ 0 & 5 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{vmatrix}.$$

Expand the determinant by the elements of the 1st column

$$D = -40 \cdot \begin{vmatrix} 6 & 2 & 0 \\ 5 & 2 & 1 \\ 0 & 3 & 1 \end{vmatrix}$$

and then subtract the elements of the 3rd row from those of the 2nd row and place 2 (the common multiplier of the elements of the 1st row) outside the determinant:

$$D = -80 \cdot \begin{vmatrix} 3 & 1 & 0 \\ 5 & -1 & 0 \\ 0 & 3 & 1 \end{vmatrix}.$$

Now expand the determinant by the elements of the 3rd column:

$$D = -80 \cdot \begin{vmatrix} 3 & 1 \\ 5 & -1 \end{vmatrix} = 640.$$

385. Find y from the system of equations

$$\begin{cases} x + 2y + 3z = 14, \\ y + 2z + 3t = 20, \\ z + 2t + 3x = 14, \\ t + 2x + 3y = 12. \end{cases}$$

Solution. Writing the system in the form

$$\begin{cases} x + 2y + 3z + 0\cdot t = 14, \\ 0\cdot x + y + 2z + 3t = 20, \\ 3x + 0\cdot y + z + 2t = 14, \\ 2x + 3y + 0\cdot z + t = 12 \end{cases}$$

we find

$$D = \begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \end{vmatrix}.$$

Now subtract the doubled elements of the 1st column from the elements of the 2nd column; the trebled elements of the 1st column from the elements of the 3rd column:

$$D = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 3 & -6 & -8 & 2 \\ 2 & -1 & -6 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ -6 & -8 & 2 \\ -1 & -6 & 1 \end{vmatrix} = (-2)\cdot(-1)\cdot \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & -1 \\ 1 & 6 & -1 \end{vmatrix}.$$

Next subtract the doubled elements of the 1st column from the elements of the 2nd column and the trebled elements of the 1st column from the elements of the 3rd column:

$$D = 2\cdot \begin{vmatrix} 1 & 0 & 0 \\ 3 & -2 & -10 \\ 1 & 4 & -4 \end{vmatrix} = 2 \begin{vmatrix} -2 & -10 \\ 4 & -4 \end{vmatrix} = 2(8 + 40) = 96.$$

Hence

$$D_y = \begin{vmatrix} 1 & 14 & 3 & 0 \\ 0 & 20 & 2 & 3 \\ 3 & 14 & 1 & 2 \\ 2 & 12 & 0 & 1 \end{vmatrix} = 2\cdot \begin{vmatrix} 1 & 7 & 3 & 0 \\ 0 & 10 & 2 & 3 \\ 3 & 7 & 1 & 2 \\ 2 & 6 & 0 & 1 \end{vmatrix}.$$

Then subtract the trebled elements of the 1st row from the elements of the 3rd row and the doubled elements of the 1st row from the elements of the 4th row:

$$D_y = 2\begin{vmatrix} 1 & 7 & 3 & 0 \\ 0 & 10 & 2 & 3 \\ 0 & -14 & -8 & 2 \\ 0 & -8 & -6 & 1 \end{vmatrix} = 2\begin{vmatrix} 10 & 2 & 3 \\ -14 & -8 & 2 \\ -8 & -6 & 1 \end{vmatrix} = 2\cdot 2\cdot 2\begin{vmatrix} 5 & 1 & 3 \\ -7 & -4 & 2 \\ -4 & -3 & 1 \end{vmatrix}.$$

Subtract the trebled elements of the 3rd row from the elements of the 1st row and the doubled elements of the 3rd row from the elements of the 2nd row:

$$D_y = 8\begin{vmatrix} 17 & 10 & 0 \\ 1 & 2 & 0 \\ -4 & -3 & 1 \end{vmatrix} = 8\begin{vmatrix} 17 & 10 \\ 1 & 2 \end{vmatrix} = 192.$$

Hence

$$y = \frac{D_y}{D} = \frac{192}{96} = 2.$$

386. Calculate the determinant

$$V = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}.$$

Solution. Subtract the 1st row multiplied by a from the 2nd row, the 2nd row multiplied by a from the 3rd row and the 3rd row multiplied by a from the 4th row:

$$V = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & c-a & d-a \\ 0 & b^2-ab & c^2-ac & d^2-ad \\ 0 & b^3-ab^2 & c^3-ac^2 & d^3-ad^2 \end{vmatrix} =$$

$$= (b-a)(c-a)(d-a)\cdot\begin{vmatrix} 1 & 1 & 1 \\ b & c & d \\ b^2 & c^2 & d^2 \end{vmatrix}.$$

Subtract the 1st row multiplied by b from the 2nd row and the 2nd row multiplied by b from the 3rd row:

$$V = (b-a)(c-a)(d-a)\cdot\begin{vmatrix} 1 & 1 & 1 \\ 0 & c-b & d-b \\ 0 & c^2-bc & d^2-db \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)(c-b)(d-b)\cdot\begin{vmatrix} 1 & 1 \\ c & d \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)(c-b)(d-b)(d-c).$$

It is easy to see that the determinant in question is equal to zero if and only if there are equal numbers among the numbers a, b, c and d.

Calculate the determinants:

387. $$\begin{vmatrix} 1 & -2 & 3 & 4 \\ 2 & 1 & -4 & 3 \\ 3 & -4 & -1 & -2 \\ 4 & 3 & 2 & -1 \end{vmatrix}.$$

388. $$\begin{vmatrix} -1 & -1 & -1 & -1 \\ -1 & -2 & -4 & -8 \\ -1 & -3 & -9 & -27 \\ -1 & -4 & -16 & -64 \end{vmatrix}.$$

389. $$\begin{vmatrix} 10 & 2 & 0 & 0 & 0 \\ 12 & 10 & 2 & 0 & 0 \\ 0 & 12 & 10 & 2 & 0 \\ 0 & 0 & 12 & 10 & 2 \\ 0 & 0 & 0 & 12 & 10 \end{vmatrix}.$$

390. $$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1-a & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1-b \end{vmatrix}.$$

Solve the following systems of equations:

391. $$\begin{cases} y - 3z + 4t = -5, \\ x - 2z + 3t = -4, \\ 3x + 2y - 5t = 12, \\ 4x + 3y - 5z = 5. \end{cases}$$

392. $$\begin{cases} x - 3y + 5z - 7t = 12, \\ 3x - 5y + 7z - t = 0, \\ 5x - 7y + z - 3t = 4, \\ 7x - y + 3z - 5t = 16. \end{cases}$$

393. $$\begin{cases} x + 2y = 5, \\ 3y + 4z = 18, \\ 5z + 6u = 39, \\ 7u + 8v = 68, \\ 9v + 10x = 55. \end{cases}$$

4.2. Linear Transformations and Matrices

By means of the equalities

$$x = a_{11}x' + a_{12}y',$$
$$y = a_{21}x' + a_{22}y'$$

the values of the variables x and y can be expressed linearly in terms of the variables x' and y'. It is customary to call these equalities the *linear transformation* of the variables x' and y'. They can also be regarded as a linear transformation of the coordinates of a point (or a vector) on a plane.

The array

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is called the *matrix* of the *linear transformation* being considered, and the determinant

$$D_A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

is called the *determinant of a linear transformation*. In what follows we shall assume $D_A \neq 0$.

We can also consider the linear transformation of three variables (that is, the transformation for space)*

$$x = a_{11}x' + a_{12}y' + a_{13}z',$$
$$y = a_{21}x' + a_{22}y' + a_{23}z',$$
$$z = a_{31}x' + a_{32}y' + a_{33}z',$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{and} \quad D_A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

are, respectively, the matrix and the determinant of that transformation.

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*Sometimes the term linear transformation is used for equalities of a more general form

$$x = a_{11}x' + a_{12}y' + a_{13}z' + b_1,$$
$$y = a_{21}x' + a_{22}y' + a_{23}z' + b_2,$$
$$z = a_{31}x' + a_{32}y' + a_{33}z' + b_3.$$

Here we consider a linear transformation for which $b_1 = b_2 = b_3 = 0$. In courses of functional analysis such a linear transformation is called a *linear operator*.

The matrix A is called *nonsingular* if $D_A \neq 0$. Now if $D_A = 0$, then the matrix A is *singular*.

The matrices

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

are known as *square matrices of the second and the third order* respectively.

For greater generality, a number of definitions will be given for matrices of the third order; they can also be easily applied to matrices of the second order.

If the elements of a square matrix satisfy the condition $a_{mn} = a_{nm}$, the matrix is *symmetrical*.

Two matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

are assumed to be *equal* ($A = B$) if and only if their respective elements are equal, i.e. if $a_{mn} = b_{mn}$ ($m, n = 1, 2, 3$).

The *sum* of two matrices A and B is a matrix specified by the equality

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{31} + b_{31} & a_{32} + b_{32} & a_{33} + b_{33} \end{pmatrix}.$$

The *product of the number m by the matrix A* is a matrix specified by the equality

$$m \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{23} \end{pmatrix} = \begin{pmatrix} ma_{11} & ma_{12} & ma_{13} \\ ma_{21} & ma_{22} & ma_{23} \\ ma_{31} & ma_{32} & ma_{33} \end{pmatrix}.$$

The *product of two matrices A and B* is symbolized as AB and is specified by the equality

$$AB = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{23} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{3} a_{1j}b_{j1} & \sum_{j=1}^{3} a_{1j}b_{j2} & \sum_{j=1}^{3} a_{1j}b_{j3} \\ \sum_{j=1}^{3} a_{2j}b_{j1} & \sum_{j=1}^{3} a_{2j}b_{j2} & \sum_{j=1}^{3} a_{2j}b_{j3} \\ \sum_{j=1}^{3} a_{3j}b_{j1} & \sum_{j=1}^{3} a_{3j}b_{j2} & \sum_{j=1}^{3} a_{3j}b_{j3} \end{pmatrix},$$

that is, *the element of the product matrix located in the ith row and the kth column is equal to the sum of the products of the corresponding elements of the ith row of the matrix A and the kth column of the matrix B.*

Matrix multiplication is not generally commutative: $AB \neq BA$.

The determinant of the product of two matrices is equal to the product of the determinants of those matrices.

A *null matrix* is a matrix whose all elements are equal to zero:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

The sum of this matrix and any matrix A is the matrix A: $A + 0 = A$. An *identity* (or unit) *matrix* is the matrix

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Multiplication of this matrix from the left or from the right by the matrix A yields the matrix A: $IA = AI = A$. A unit matrix is associated with the identity linear transformation

$$x = x', \quad y = y', \quad z = z'.$$

Every nonsingular ($D_A \neq 0$) square matrix A possesses an inverse.

A matrix B is *inverse* to the matrix A if the product AB is equal to a unit matrix: $AB = I$.

A matrix inverse to A is usually designated as A^{-1}.

The inverse of a matrix can be found by the formula

$$A^{-1} = \begin{pmatrix} A_{11}/D_A & A_{21}/D_A & A_{31}/D_A \\ A_{12}/D_A & A_{22}/D_A & A_{32}/D_A \\ A_{13}/D_A & A_{23}/D_A & A_{33}/D_A \end{pmatrix},$$

where A_{mn} is a *cofactor* of the element a_{mn} of the matrix in its determinant, that is,

the second-order minor, obtained by deleting the mth row and the nth column in the determinant of the matrix A, multiplied by $(-1)^{m+n}$.

Multiplication of the matrices A and A^{-1} is commutative, i.e. $AA^{-1} = A^{-1}A = I$.

The name *column matrix* is used for the matrix

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

The product AX is specified by the equality

$$AX = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}.$$

The system of equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

can be written in the form $AX = B$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

The solution of this system is of the form $X = A^{-1}B$ (if $D_A \neq 0$).

The *characteristic equation* of the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is the equation

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0.$$

The roots $\lambda_1, \lambda_2, \lambda_3$ of this equation are called the *characteristic numbers* of the matrix; they are always real if the original matrix is symmetrical.

The system of equations

$$\begin{cases} (a_{11} - \lambda)\xi_1 + a_{12}\xi_2 + a_{13}\xi_3 = 0, \\ a_{21}\xi_1 + (a_{22} - \lambda)\xi_2 + a_{23}\xi_3 = 0, \\ a_{31}\xi_1 + a_{32}\xi_2 + (a_{33} - \lambda)\xi_3 = 0, \end{cases}$$

in which λ has one of the values $\lambda_1, \lambda_2, \lambda_3$ and whose determinant is zero due to this fact, specifies the triple of numbers (ξ_1, ξ_2, ξ_3) corresponding to the given characteristic number.

This set of three numbers (ξ_1, ξ_2, ξ_3) defines the vector $\mathbf{r} = \xi_1\mathbf{i} + \xi_2\mathbf{j} + \xi_3\mathbf{k}$ known as the *eigenvector* of the matrix.

394. Find the sum of the matrices

$$A = \begin{pmatrix} 3 & 5 & 7 \\ 2 & -1 & 0 \\ 4 & 3 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -2 \\ -1 & 0 & 1 \end{pmatrix}.$$

Solution. We have

$$A + B = \begin{pmatrix} 3+1 & 5+2 & 7+4 \\ 2+2 & -1+3 & 0-2 \\ 4-1 & 3+0 & 2+1 \end{pmatrix} = \begin{pmatrix} 4 & 7 & 11 \\ 4 & 2 & -2 \\ 3 & 3 & 3 \end{pmatrix}.$$

395. Find the matrix $2A + 5B$ if

$$A = \begin{pmatrix} 3 & 5 \\ 4 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}.$$

Solution. We have

$$2A = \begin{pmatrix} 6 & 10 \\ 8 & 2 \end{pmatrix}, \quad 5B = \begin{pmatrix} 10 & 15 \\ 5 & -10 \end{pmatrix}, \quad 2A + 5B = \begin{pmatrix} 16 & 25 \\ 13 & -8 \end{pmatrix}.$$

396. Find the products AB and BA of the matrices if

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & 4 \\ 1 & 2 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \\ 3 & 2 & 1 \end{pmatrix}.$$

Solution. We have

$$AB = \begin{pmatrix} 1\cdot2 + 3\cdot1 + 1\cdot3 & 1\cdot1 + 3(-1) + 1\cdot2 & 1\cdot0 + 3\cdot2 + 1\cdot1 \\ 2\cdot2 + 0\cdot1 + 4\cdot3 & 2\cdot1 + 0(-1) + 4\cdot2 & 2\cdot0 + 0\cdot2 + 4\cdot1 \\ 1\cdot2 + 2\cdot1 + 3\cdot3 & 1\cdot1 + 2(-1) + 3\cdot2 & 1\cdot0 + 2\cdot2 + 3\cdot1 \end{pmatrix} =$$

$$= \begin{pmatrix} 8 & 0 & 7 \\ 16 & 10 & 4 \\ 13 & 5 & 7 \end{pmatrix},$$

$$BA = \begin{pmatrix} 2\cdot1 + 1\cdot2 + 0\cdot1 & 2\cdot3 + 1\cdot0 + 0\cdot2 & 2\cdot1 + 1\cdot4 + 0\cdot3 \\ 1\cdot1 - 1\cdot2 + 2\cdot1 & 1\cdot3 - 1\cdot0 + 2\cdot2 & 1\cdot1 - 1\cdot4 + 2\cdot3 \\ 3\cdot1 + 2\cdot2 + 1\cdot1 & 3\cdot3 + 2\cdot0 + 1\cdot2 & 3\cdot1 + 2\cdot4 + 1\cdot3 \end{pmatrix} =$$

$$= \begin{pmatrix} 4 & 6 & 6 \\ 1 & 7 & 3 \\ 8 & 11 & 14 \end{pmatrix},$$

397. Find A^3 if $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$.

Solution. We find

$$A^2 = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 9+2 & 6+8 \\ 3+4 & 2+16 \end{pmatrix} = \begin{pmatrix} 11 & 14 \\ 7 & 18 \end{pmatrix},$$

$$A^3 = A^2\cdot A = \begin{pmatrix} 11 & 14 \\ 7 & 18 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 33+14 & 22+56 \\ 21+18 & 14+72 \end{pmatrix} = \begin{pmatrix} 47 & 78 \\ 39 & 86 \end{pmatrix}.$$

398. Given the linear transformation $x = x' + y' + z'$, $y = x' + y'$, $z = x'$ and the points in the system of coordinates x', y', z': $(1; -1; 1)$, $(3; -2; -1)$, $(-1; -2; -3)$. Determine the coordinates of these points in the x, y, z system.

Solution. Substituting the coordinates of the points into the equalities specifying the given linear transformation, we get the following: if $x' = 1, y' = -1, z' = 1$, then $x = 1$, $y = 0$, $z = 1$, i.e. $(1; 0: 1)$; if $x' = 3$, $y' = -2$, $z' = -1$, then $x = 0$, $y = 1$, $z = 3$, i.e. $(0; 1; 3)$; if $x' = -1$, $y' = -2$, $z' = -3$, then $x = -6, y = -3, z = -1$, i.e. $(-6; -3; -1)$.

399. Write the linear transformation of the previous problem for transition from the coordinates x, y, z to the coordinates x', y', z'.

Solution. We have $x' = z$ (from the third equality); $y' = y - z$ (we subtract the third equality from the second); $z' = x - y$ (we subtract the second equality from the first).

400. Given the linear transformation $x = x' + 2y'$, $y = 3x' + 4y'$. The coordinates of what points does it leave unchanged?

Solution. It is necessary to find x and y if $x = x'$, $y = y'$, i.e. $x = x + 2y$, $y = 3x + 4y$. Consequently, $x = x' = 0, y = y' = 0$.

401. The coordinates of what points does the linear transformation $x = 3x' - 2y'$, $y = 5x' - 4y'$ leave unchanged?

Solution. We have $x = 3x - 2y$, $y = 5x - 4y$. Consequently, $x = y = x' = y'$, that is, the linear transformation does not change the coordinates of the points $(t; t)$ with the same coordinates.

402. Find the value of the matrix polynomial $2A^2 + 3A + 5I$ for $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 4 & 1 & 1 \end{pmatrix}$ if I is an identity matrix of the third order.

Solution. We have

$$A^2 = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 4 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 4 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 6 & 5 \\ 8 & 11 & 6 \\ 9 & 8 & 10 \end{pmatrix}, \quad 2A^2 = \begin{pmatrix} 20 & 12 & 10 \\ 16 & 22 & 12 \\ 18 & 16 & 20 \end{pmatrix},$$

$$3A = \begin{pmatrix} 3 & 3 & 6 \\ 3 & 9 & 3 \\ 12 & 3 & 3 \end{pmatrix}, \quad 5I = 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix},$$

$$2A^2 + 3A + 5I = \begin{pmatrix} 28 & 15 & 16 \\ 19 & 36 & 15 \\ 30 & 19 & 28 \end{pmatrix}.$$

403. Given the two linear transformations $x = a_{11}x' + a_{12}y'$, $y = a_{21}x' + a_{22}y'$ and $x' = b_{11}x'' + b_{12}y''$, $y' = b_{21}x'' + b_{22}y''$. Substituting x' and y' from the second transformation into the first, we obtain a linear transformation expressing x and y in terms of x'' and y''. Show that the matrix of the transformation obtained is equal to the product of the matrices of the first and second transformations.

Solution. We have

$$x = a_{11}(b_{11}x'' + b_{12}y'') + a_{12}(b_{21}x'' + b_{22}y'') =$$
$$= (a_{11}b_{11} + a_{12}b_{21})x'' + (a_{11}b_{12} + a_{12}b_{22})y'',$$
$$y = a_{21}(b_{11}x'' + b_{12}y'') + a_{22}(b_{21}x'' + b_{22}y'') =$$
$$= (a_{21}b_{11} + a_{22}b_{21})x'' + (a_{21}b_{12} + a_{22}b_{22})y''.$$

The matrix of the linear transformation obtained is of the form

$$\begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix},$$

that is, it is the product of the matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$.

404. Given the matrix $\begin{pmatrix} 5 & 2 \\ 4 & 3 \end{pmatrix}$. Find its characteristic numbers and eigenvectors.

Solution. We set up the characteristic equation $\begin{vmatrix} 5-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$, or $(5-\lambda)(3-\lambda) - 8 = 0$, i.e. $\lambda^2 - 8\lambda + 7 = 0$; the characteristic numbers are $\lambda_1 = 1$, $\lambda_2 = 7$. The eigenvector corresponding to the first characteristic number can be found from the system of equations

$$\begin{cases} (5-\lambda_1)\xi_1' + 2\xi_2' = 0, \\ 4\xi_1' + (3-\lambda_1)\xi_2' = 0; \end{cases}$$

since $\lambda_1 = 1$, it follows that ξ_1' and ξ_2' are connected by the relation $2\xi_1' + \xi_2' = 0$.

Putting $\xi_1' = \alpha$ (α being an arbitrary number), we get $\xi_2' = -2\alpha$ and the eigenvector corresponding to the characteristic number $\lambda_1 = 1$ is $\mathbf{r}_1 = \alpha\mathbf{i} - 2\alpha\mathbf{j}$.

Next we find the second eigenvector. We have

$$\begin{cases} (5 - \lambda_2)\xi_1'' + 2\xi_2'' = 0, \\ 4\xi_1'' + (3 - \lambda_2)\xi_2'' = 0. \end{cases}$$

Substituting the value $\lambda_2 = 7$, we arrive at a relation $\xi_1'' - \xi_2'' = 0$, i.e. $\xi_1'' = \xi_2'' = \beta$. The eigenvector corresponding to the second characteristic number is $\mathbf{r}_2 = \beta\mathbf{i} + \beta\mathbf{j}$.

405. Find the characteristic numbers and the eigenvectors of the matrix

$$\begin{Vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{Vmatrix}.$$

Solution. We set up the characteristic equation

$$\begin{vmatrix} 3 - \lambda & -1 & 1 \\ -1 & 5 - \lambda & -1 \\ 1 & -1 & 3 - \lambda \end{vmatrix} = 0,$$

or

$$(3 - \lambda)[(5 - \lambda)(3 - \lambda) - 1] + (-3 + \lambda + 1) + (1 - 5 + \lambda) = 0.$$

Simple transformations reduce the equation to the form $(3 - \lambda)(\lambda^2 - 8\lambda + 12) = 0$, whence $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 6$.

Next we find the eigenvector corresponding to the characteristic number $\lambda_1 = 2$. From the system of equations

$$\begin{cases} \xi_1' - \xi_2' + \xi_3' = 0, \\ -\xi_1' + 3\xi_2' - \xi_3' = 0, \\ \xi_1' - \xi_2' + \xi_3' = 0 \end{cases}$$

(one of the equations of the system is a corollary of the other two and can be discarded), we find $\xi_2' = 0$, $\xi_3 = -\xi_1'$. We put $\xi_1' = \alpha$, then $\xi_2' = 0$, $\xi_3' = -\alpha$ and $\mathbf{r}_1 = \alpha\mathbf{i} - \alpha\mathbf{k}$.

Now we find the eigenvector corresponding to the value $\lambda_2 = 3$. We obtain a system of equations

$$\begin{cases} -\xi_2'' + \xi_3'' = 0, \\ -\xi_1'' + 2\xi_2'' - \xi_3'' = 0, \\ \xi_1'' - \xi_2'' = 0 \end{cases}$$

(one of these equations is a corollary of the other two). Hence we have $\xi_1'' = \xi_2'' = \xi_3''' = \beta$ and $\mathbf{r}_2 = \beta\mathbf{i} + \beta\mathbf{j} + \beta\mathbf{k}$.

We find the eigenvector corresponding to the value $\lambda_3 = 6$. We set a system of equations

$$\begin{cases} -3\xi_1''' - \xi_2''' + \xi_3''' = 0, \\ -\xi_1''' - \xi_2''' - \xi_3''' = 0, \\ \xi_1''' - \xi_2''' - 3\xi_3''' = 0 \end{cases}$$

(again one of the equations is a corollary of the other two). Solving the system, we find $\xi_1''' = \gamma$, $\xi_2''' = -2\gamma$, $\xi_3''' = \gamma$ and $\mathbf{r}_3 = \gamma\mathbf{i} - 2\gamma\mathbf{j} + \gamma\mathbf{k}$.

Thus we see that the eigenvectors of the given matrix have the form $\mathbf{r}_1 = \alpha(\mathbf{i} - \mathbf{k})$; $\mathbf{r}_2 = \beta(\mathbf{i} + \mathbf{j} + \mathbf{k})$; $\mathbf{r}_3 = \gamma(\mathbf{i} - 2\mathbf{j} - \mathbf{k})$, where α, β, γ are arbitrary nonzero numbers.

406. Given the matrix $A = \begin{pmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{pmatrix}$. Find its inverse.

Solution. We calculate the determinant of the matrix A:

$$D_A = \begin{vmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{vmatrix} = 27 + 2 - 24 = 5.$$

Then we find the cofactors of the elements of the determinant:

$$A_{11} = \begin{vmatrix} 3 & 1 \\ 3 & 4 \end{vmatrix} = 9, \quad A_{21} = -\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = -2, \quad A_{31} = \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} = -4,$$

$$A_{12} = -\begin{vmatrix} 1 & 1 \\ 5 & 4 \end{vmatrix} = 1, \quad A_{22} = \begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix} = 2, \quad A_{32} = -\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = -1,$$

$$A_{13} = \begin{vmatrix} 1 & 3 \\ 5 & 3 \end{vmatrix} = -12, \quad A_{23} = -\begin{vmatrix} 3 & 2 \\ 5 & 3 \end{vmatrix} = 1, \quad A_{33} = \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} = 7.$$

Consequently,

$$A^{-1} = \begin{pmatrix} 9/5 & -2/5 & -4/5 \\ 1/5 & 2/5 & -1/5 \\ -12/5 & 1/5 & 7/5 \end{pmatrix}.$$

407. Solve the following system of equations:

$$\begin{cases} 2x + 3y + 2z = 9, \\ x + 2y - 3z = 14, \\ 3x + 4y + z = 16, \end{cases}$$

representing it as a matrix equation.

Solution. Let us rewrite the system in the form $AX = B$, where

$$A = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & -3 \\ 3 & 4 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 9 \\ 14 \\ 16 \end{pmatrix}.$$

The solution of the matrix equation has the form $X = A^{-1}B$. Let us find A^{-1}. We have

$$D_A = \begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & -3 \\ 3 & 4 & 1 \end{vmatrix} = 28 - 30 - 4 = -6.$$

Let us now calculate the cofactors of the elements of the determinant:

$$A_{11} = \begin{vmatrix} 2 & -3 \\ 4 & 1 \end{vmatrix} = 14, \quad A_{21} = -\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 5,$$

$$A_{31} = \begin{vmatrix} 3 & 2 \\ 2 & -3 \end{vmatrix} = -13, \quad A_{12} = -\begin{vmatrix} 1 & -3 \\ 3 & 1 \end{vmatrix} = -10,$$

$$A_{22} = \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} = -4, \quad A_{32} = -\begin{vmatrix} 2 & 2 \\ 1 & -3 \end{vmatrix} = 8,$$

$$A_{13} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2, \quad A_{23} = -\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = 1,$$

$$A_{33} = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1.$$

Thus we have

$$A^{-1} = -\frac{1}{6}\begin{pmatrix} 14 & 5 & -13 \\ -10 & -4 & 8 \\ -2 & 1 & 1 \end{pmatrix},$$

whence

$$X = -\frac{1}{6}\begin{pmatrix} 14 & 5 & -13 \\ -10 & -4 & 8 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 9 \\ 14 \\ 16 \end{pmatrix} = -\frac{1}{6}\begin{pmatrix} 126 + 70 - 208 \\ -90 - 56 + 128 \\ 18 + 14 + 16 \end{pmatrix}$$

$$= -\frac{1}{6}\begin{pmatrix} -12 \\ -18 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}.$$

Consequently, $x = 2, y = 3, z = -2$.

408. Normalize the vector $\mathbf{x} = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$.

Solution. To normalize the vector $\mathbf{x} = \xi_1\mathbf{i} + \xi_2\mathbf{j} + \xi_3\mathbf{k}$ means to find a unit vector of the same direction. The vector

$$\mathbf{x}_0 = (\xi_1\mathbf{i} + \xi_2\mathbf{j} + \xi_3\mathbf{k})/\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}.$$

is just such a vector. In the given case $\mathbf{x}_0 = (3/13)\mathbf{i} + (4/13)\mathbf{j} + (12/13)\mathbf{k}$.

4.3. Reducing General Equations of Second-Order Curves and Surfaces to the Canonical Form

Expressions of the form

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2$$

and

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz$$

are known as *quadratic forms* of two and three variables respectively.

Symmetric matrices

$$A_2^{(2)} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \text{where } a_{21} = a_{12}$$

and

$$A_3^{(3)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \text{where } a_{21} = a_{12},\ a_{31} = a_{13} \text{ and } a_{32} = a_{23}$$

are the *matrices of these forms.*

By means of linear transformations of variables quadratic forms can be reduced to the forms which do not contain products of the new variables (are reduced, as they say, to the algebraic sum of the squares); in other words, a quadratic form of two variables can be reduced to the form $\lambda_1 x'^2 + \lambda_2 y'^2$, and a quadratic form of three variables, to the form $\lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2$. In this case the coefficients λ_1, λ_2 and λ_3 are the characteristic numbers of the matrices of the respective forms.

The requisite linear transformation of variables can be found as follows: a triple (or a pair, for a quadratic form of two variables) of normalized, pairwise orthogonal eigenvectors is determined, corresponding to the characteristic numbers $\lambda_1, \lambda_2, \lambda_3$:

$$\begin{aligned} \mathbf{e}_1 &= \alpha_1 \mathbf{i} + \beta_1 \mathbf{j} + \gamma_1 \mathbf{k}, \\ \mathbf{e}_2 &= \alpha_2 \mathbf{i} + \beta_2 \mathbf{j} + \gamma_2 \mathbf{k}, \\ \mathbf{e}_3 &= \alpha_3 \mathbf{i} + \beta_3 \mathbf{j} + \gamma_3 \mathbf{k}. \end{aligned}$$

Since the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are normalized and orthogonal, the following identities must be satisfied:

$$\alpha_i^2 + \beta_i^2 + \gamma_i^2 = 1, \quad i = 1, 2, 3; \quad \alpha_i\alpha_j + \beta_i\beta_j + \gamma_i\gamma_j = 0, \quad i, j = 1, 2, 3, i \neq j.$$

Then the matrix of the transformation of variables has the form

$$S = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix};$$

in other words, we must assume

$$x = \alpha_1 x' + \alpha_2 y' + \alpha_3 z',$$
$$y = \beta_1 x' + \beta_2 y' + \beta_3 z',$$
$$z = \gamma_1 x' + \gamma_2 y' + \gamma_3 z'$$

(for the case of two variables all the formulas are correspondingly simplified). This is a *linear orthogonal transformation*: in this case the determinant of the matrix S is equal to ± 1: $D_S = \pm 1$.

A linear orthogonal transformation is used to reduce to the canonical form a general equation of a curve or a surface of the second order. If the mutual orientation of the new coordinate axes has to be retained in this case, then an auxiliary condition $D_S = 1$ is imposed on the matrix of transformation S.

Transformation of the equation of a second-order curve or surface to the canonical form is carried out as follows:

(a) a linear orthogonal transformation of coordinates is found which reduces the quadratic form of the leading terms of the equation of a curve or a surface to the sum of the squares and a requisite change is performed in the equation. As a result of this transformation the terms with the products of coordinates are removed from the equation;

(b) performing after that a parallel translation of the new coordinate axes (in space it is sometimes also necessary to make an additional rotation of two axes in one of the coordinate planes), the equation is reduced to the required canonical form.

409. Reduce to the canonical form the equation of the curve

$$5x^2 + 4xy + 8y^2 - 32x - 56y + 80 = 0.$$

Solution. The matrix of the leading terms is $A = \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}$. We set up the characteristic equation of the matrix:

$$\begin{vmatrix} 5-\lambda & 2 \\ 2 & 8-\lambda \end{vmatrix} = 0, \quad \text{i.e.} \quad \lambda^2 - 13\lambda + 36 = 0.$$

Next we find the characteristic numbers $\lambda_1 = 4$, $\lambda_2 = 9$. Putting $\lambda_1 = 4$, we obtain the following system of equations for determining the corresponding eigenvector:

$$\begin{cases} \xi_1 + 2\xi_2 = 0, \\ 2\xi_1 + 4\xi_2 = 0. \end{cases}$$

Hence we have $\xi_1 = -2\xi_2$; assuming $\xi_2 = -\alpha$, we find $\xi_1 = 2\alpha$ and $\mathbf{r}_1 = \alpha(2\mathbf{i} - \mathbf{j})$. We normalize the vector $\mathbf{r}_1$:

$$\mathbf{e}_1 = (2/\sqrt{5})\mathbf{i} - (1/\sqrt{5})\mathbf{j}.$$

Putting $\lambda_2 = 9$, we obtain the following system of equations to determine the second eigenvector:

$$\begin{cases} -4\eta_1 + 2\eta_2 = 0, \\ 2\eta_1 - \eta_2 = 0. \end{cases}$$

Hence $\eta_2 = 2\eta_1$ and $\mathbf{r}_2 = \beta(\mathbf{i} + 2\mathbf{j})$. As a result of normalization, we obtain

$$\mathbf{e}_2 = (1/\sqrt{5})\mathbf{i} + (2/\sqrt{5})\mathbf{j}.$$

The vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal: $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$.

We shall now use the normalized orthogonal eigenvectors to construct a matrix of transformation of coordinates

$$S = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}; \quad D_S = 1.$$

From this we get

$$x = (2/\sqrt{5})x' + (1/\sqrt{5})y', \quad y = (-1/\sqrt{5})x' + (2/\sqrt{5})y'.$$

Substituting the expressions for x and y we have found into the equation of the curve

$$5\left(\frac{2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y'\right)^2 + 4\left(\frac{2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y'\right)\left(-\frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}y'\right) +$$
$$+ 8\left(-\frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}y'\right)^2 - 32\left(\frac{2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y'\right) -$$
$$- 56\left(\frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}y'\right) + 80 = 0,$$

we get, after removing the parentheses and collecting the like terms,

$$4x'^2 + 9y'^2 - \frac{8}{\sqrt{5}}x' - \frac{144}{\sqrt{5}}y' + 80 = 0.$$

Note that in the transformed equation the coefficients in x'^2 and y'^2 turned out to be (as should have been expected) the characteristic numbers λ_1 and λ_2. Let us rewrite the equation in the form

$$4\left(x'^2 - \frac{2}{\sqrt{5}}x'\right) + 9\left(y'^2 - \frac{16}{\sqrt{5}}y'\right) + 80 = 0.$$

Let us now complete the expressions in parentheses to perfect squares:

$$4\left(x'^2 - \frac{2}{\sqrt{5}}x' + \frac{1}{5} - \frac{1}{5}\right) + 9\left(y'^2 - \frac{16}{\sqrt{5}}y' + \frac{64}{5} - \frac{64}{5}\right) + 80 = 0,$$

or

$$4\left(x' - \frac{1}{\sqrt{5}}\right)^2 - \frac{4}{5} + 9\left(y' - \frac{8}{\sqrt{5}}\right)^2 - \frac{576}{5} + 80 = 0.$$

The final result is

$$4\left(x' - \frac{1}{\sqrt{5}}\right)^2 + 9\left(y' - \frac{8}{\sqrt{5}}\right)^2 = 36.$$

Now we perform a parallel translation of the coordinate axes putting $x'' = x' - 1/\sqrt{5}$, $y'' = y' - 8/\sqrt{5}$, and get

$$4x''^2 + 9y''^2 = 36, \quad \text{or} \quad x''^2/9 + y''^2/4 = 1$$

(canonical equation of an ellipse).

410. Reduce to the canonical form the equation of the curve

$$9x^2 + 24xy + 16y^2 - 230x + 110y - 225 = 0.$$

Solution. The characteristic equation has the form

$$\begin{vmatrix} 9 - \lambda & 12 \\ 12 & 16 - \lambda \end{vmatrix} = 0, \quad \text{or} \quad \lambda^2 - 25\lambda = 0, \quad \text{i.e.} \quad \lambda_1 = 0, \quad \lambda_2 = 25.$$

At $\lambda = 0$, we get the system

$$\begin{cases} 9\xi_1 + 12\xi_2 = 0, \\ 12\xi_1 + 16\xi_2 = 0. \end{cases}$$

Each of these equations can be reduced to the equation $\xi_1/4 = \xi_2/(-3)$. Consequently, the vector $\mathbf{r} = \alpha(4\mathbf{i} - 3\mathbf{j})$ is the eigenvector of the matrix, and at $\alpha = 1/5$ we find the normalized eigenvector $\mathbf{e}_1 = (4/5)\mathbf{i} - (3/5)\mathbf{j}$.

At $\lambda = 25$, we get the system

$$\begin{cases} -16\eta_1 + 12\eta_2 = 0, \\ 12\eta_1 - 9\eta_2 = 0. \end{cases}$$

In an analogous way, we get from this system the second normalized eigenvector $\mathbf{e}_2 = (3/5)\mathbf{i} + (4/5)\mathbf{j}$ $(\mathbf{e}_1 \cdot \mathbf{e}_2 = 0)$.

The matrix of transformation of coordinates has the form

$$S = \begin{pmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{pmatrix} \quad (D_S = 1);$$

the transformation formulas are $x = (4/5)x' + (3/5)y'$, $y = (-3/5)x' + (4/5)y'$.

Having rewritten the equation of the curve in the form

$$(3x + 4y)^2 - 230x + 110y - 225 = 0,$$

we pass over to new coordinates:

$$25y'^2 - 230\left(\frac{4}{5}x' + \frac{3}{5}y'\right) + 110\left(-\frac{3}{5}x' + \frac{4}{5}y'\right) - 225 = 0.$$

After collecting the like terms and cancelling by 25, we arrive at the equation

$$y'^2 - 10x' - 2y' - 9 = 0.$$

It can be rewritten as $(y' - 1)^2 = 10(x' + 1)$. Accepting the point $O'(-1; 1)$ as the new origin, and performing a parallel translation of the axes, we arrive at the canonical equation of the given curve $y''^2 = 10x''$ (a parabola).

411. Reduce to the canonical form the equation of a surface

$$3x^2 + 5y^2 + 3z^2 - 2xy - 2xz - 2yz - 12x - 10 = 0.$$

Solution. Here the matrix of the leading terms of the equation of the surface has the form

$$\begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}.$$

The characteristic numbers of the matrix are determined by the equation

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0,$$

which can be reduced to the form $(3 - \lambda)(\lambda^2 - 8\lambda + 12) = 0$; from this we find $\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 6$.

At $\lambda = 2$ we get the system

$$\begin{cases} u_1 - u_2 + u_3 = 0, \\ -u_1 + 3u_2 - u_3 = 0, \\ u_1 - u_2 + u_3 = 0. \end{cases}$$

The indicated value of λ is associated with the eigenvector $(\alpha; 0; -\alpha)$. Normalization leads to the vector $\mathbf{e}_1 = (1/\sqrt{2})\mathbf{i} - (1/\sqrt{2})\mathbf{k}$.

At $\lambda = 3$, we obtain the system

$$\begin{cases} -v_2 + v_3 = 0, \\ -v_1 + 2v_2 - v_3 = 0, \\ v_1 - v_2 = 0. \end{cases}$$

From this we find the second normalized eigenvector $\mathbf{e}_2 = (1/\sqrt{3})\mathbf{i} + (1/\sqrt{3})\mathbf{j} + (1/\sqrt{3})\mathbf{k}$. The vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal: $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$.

At $\lambda = 6$, we get the system

$$\begin{cases} -3w_1 - w_2 + w_3 = 0, \\ -w_1 - w_2 - w_3 = 0, \\ w_1 - w_2 - 3w_3 = 0. \end{cases}$$

The corresponding normalized eigenvector (the third) is the vector $\mathbf{e}_3 = (1/\sqrt{6})\mathbf{i} - (2/\sqrt{6})\mathbf{j} + (1/\sqrt{6})\mathbf{k}$, which is orthogonal to the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$: $\mathbf{e}_1 \cdot \mathbf{e}_3 = 0$, $\mathbf{e}_2 \cdot \mathbf{e}_3 = 0$. Now we find the matrix of transformation of coordinates:

$$S = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix}.$$

This matrix is used to obtain the formulas for transformation of coordinates:

$$x = (1/\sqrt{2})x' + (1/\sqrt{3})y' + (1/\sqrt{6})z', \quad y = (1/\sqrt{3})y' - (2/\sqrt{6})z',$$
$$z = (-1/\sqrt{2})x' + (1/\sqrt{3})y' + (1/\sqrt{6})z'.$$

Substituting the expressions for x, y and z into the equation of the surface, we obtain after simplifications

$$2x'^2 + 3y'^2 + 6z'^2 - 6\sqrt{2}x' - 4\sqrt{3}y' - 2\sqrt{6}x' - 10 = 0.$$

As should be expected, the coefficients in x'^2, y'^2, z'^2 are the numbers λ_1, λ_2, λ_3 respectively. Let us rewrite the equation as

$$2\left(x'^2 - \frac{6}{\sqrt{2}}x'\right) + 3\left(y'^2 - \frac{4}{\sqrt{3}}y'\right) + 6\left(z'^2 - \frac{2}{\sqrt{6}}z'\right) = 10,$$

which yields, after completion of the expressions in parentheses to perfect squares,

$$2\left(x' - \frac{3}{\sqrt{2}}\right)^2 + 3\left(y' - \frac{2}{\sqrt{3}}\right)^2 + 6\left(z' - \frac{1}{\sqrt{6}}\right)^2 = 24.$$

Making a parallel translation of the coordinate axes by the formulas $x' = x'' + 3/\sqrt{2}$, $y' = y'' + 2/\sqrt{3}$, $z' = z'' + 1/\sqrt{6}$ and dividing the equation by 24, we arrive at the canonical equation of an ellipsoid $x''^2/12 + y''^2/8 + z''^2/4 = 1$.

412. Given the matrix $A = \begin{pmatrix} 5 & 8 & 4 \\ 3 & 2 & 5 \\ 7 & 6 & 0 \end{pmatrix}$. What matrix B should be added to the matrix A to obtain a unit matrix?

413. Given the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Find the sum of the matrices $A^2 + A + I$.

414. Given the matrix $A = \begin{pmatrix} 10 & 20 & -30 \\ 0 & 10 & 20 \\ 0 & 0 & 10 \end{pmatrix}$. Find its inverse.

415. Given two linear transformations

$$x = a_{11}x' + a_{12}y' + a_{13}z', \quad y = a_{21}x' + a_{22}y' + a_{23}z', \quad z = a_{31}x' + a_{32}y' + a_{33}z';$$

$$x' = b_{11}x'' + b_{12}y'' + b_{13}z'', \quad y' = b_{21}x'' + b_{22}y'' + b_{23}z'', \quad z' = b_{31}x'' + b_{32}y'' + b_{33}z''.$$

Substituting x', y' and z' from the second transformation into the first, we obtain a linear transformation expressing x, y, z in terms of x'', y'', z''. Show that the matrix of the transformation obtained is equal to the product of the matrices of the first and second transformations.

416. Find the characteristic numbers and the normalized eigenvectors of the matrix $\begin{pmatrix} 7 & 4 \\ 5 & 6 \end{pmatrix}$.

417. Find the characteristic numbers and the eigenvectors of the matrix $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$.

418. Reduce the equation of the curve $5x^2 + 6xy + 5y^2 - 16x - 16y = 0$ to the canonical form.

419. Solve the system of equations

$$\begin{cases} 3x + 4y = 11, \\ 5y + 6z = 28, \\ x + 2z = 7, \end{cases}$$

representing it as a matrix equation.

420. Reduce to the canonical form the equation of the curve $7x^2 + 16xy - 23y^2 - 14x - 16y - 218 = 0$.

421. Reduce to the canonical form the equation of the curve $x^2 + 2xy + y^2 - 8x + 4 = 0$.

422. Reduce to the canonical form the equation of the surface $x^2 + 5y^2 + z^2 + 2xy + 6xz + 2yz - 6 = 0$.

Hint. The formulas for transformation of coordinates are

$$x = (1/\sqrt{3})x' + (1/\sqrt{6})y' + (1/\sqrt{2})z',$$

$$y = -(1/\sqrt{3})x' + (2/\sqrt{6})y', \quad z = (1/\sqrt{3})x' + (1/\sqrt{6})y' - (1/\sqrt{2})z'.$$

423. Reduce to the canonical form the equation of the surface

$$2x^2 + y^2 + 2z^2 - 2xy - 2yz + x - 4y - 3z + 2 = 0.$$

Hint. The formulas for transformation of coordinates are

$$x = -(1/\sqrt{6})x' - (1/\sqrt{2})y' + (1/\sqrt{3})z', \quad x' = x'';$$

$$y = -(2/\sqrt{6})x' - (1/\sqrt{3})z', \quad y' = y'' + 1/\sqrt{2};$$
$$z = -(1/\sqrt{6})x' + (1/\sqrt{2})y' + (1/\sqrt{3})z', \quad z' = z'' + 1/\sqrt{3}.$$

424. Given the linear transformation $x = 6x' + y' - 2z'$, $y = -18x' + 2y' + 6z'$, $z = 2x' + 2y'$. The coordinates of what points are doubled as a result of this transformation?

425. Given two linear transformations:

$$x = x' + y' + 2z', \quad y = x' + 2y' + 6z', \quad z = 2x' + 3y';$$
$$x = 2x' + 2z', \; y = x' + 3y' + 4z', \quad z = x' + 3y' + 2z'.$$

Find the points for which these transformations give the same result.

426. Find the points whose coordinates do not change under the linear transformation $x = x'\cos\alpha - y'\sin\alpha$, $y = x'\sin\alpha + y'\cos\alpha$.

427. Find the set of points whose coordinates interchange their places under the linear transformation $x = x'\cos\alpha - y'\sin\alpha$, $y = x'\sin\alpha + y'\cos\alpha$.

4.4. Rank of a Matrix. Equivalent Matrices

Given the rectangular matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdots & \cdots \\ a_{m1} & a_{m2} \cdots a_{mn} \end{pmatrix}.$$

Let us isolate k arbitrary rows and k arbitrary columns ($k \leqslant m$, $k \leqslant n$) in this matrix. A determinant of the kth order composed from the elements of the matrix A located at the intersection of the isolated rows and columns is known as a *minor* of the kth order of the matrix A. The matrix A possesses $C_m^k \cdot C_n^k$ minors of the kth order.

Let us consider various nonzero minors of the matrix A. The *rank* of the matrix A is the greatest order of the nonzero minor of that matrix. Now if all the elements of the matrix are equal to zero, then the rank of the matrix is taken to be zero.

Every nonzero minor of a matrix whose order is equal to the rank of that matrix is called the *base minor* of that matrix.

The rank of the matrix A will be designated as $r(A)$. If $r(A) = r(B)$, the matrices A and B are said to be *equivalent*. In this case it is customary to write $A \sim B$.

It is useful to bear in mind that the rank of a matrix does not change as a result of *elementary transformations*. The following transformations are usually understood as elementary:

(1) replacement of rows by columns and columns by the respective rows;
(2) interchange of the rows of a matrix;
(3) deletion of the row whose all elements are zero;
(4) multiplication of some row by a nonzero number;
(5) addition to the elements of one row the respective elements of another row.

428. Determine the rank of the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{pmatrix}$.

Solution. All the second- and third-order minors of the given matrix are equal to zero since the elements of the rows of these minors are proportional, while the first-order minors (the elements of the matrix themselves) are nonzero. Consequently, the rank of the matrix is equal to unity.

429. Determine the rank of the matrix $\begin{pmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 11 \end{pmatrix}$.

Solution. Deleting the 2nd row, and then the 2nd, 3rd and 4th columns from the matrix, we obtain the martix $\begin{pmatrix} 1 & 5 \\ 2 & 11 \end{pmatrix}$, equivalent to the given matrix. Since $\begin{vmatrix} 1 & 5 \\ 2 & 11 \end{vmatrix} = 1 \neq 0$, the rank of the given matrix is equal to 2.

430. Determine the rank of the matrix $\begin{pmatrix} 3 & 5 & 7 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}$.

Solution. Add up the elements of the 1st and the 3rd row respectively:

$$\begin{pmatrix} 3 & 5 & 7 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix} \sim \begin{pmatrix} 4 & 8 & 12 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}.$$

Then divide by 4 the elements of the first row:

$$\begin{pmatrix} 3 & 5 & 7 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}.$$

From the elements of the 1st row subtract the corresponding elements of the 2nd row:

$$\begin{pmatrix} 3 & 5 & 7 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}.$$

Then delete the first row:

$$\begin{pmatrix} 3 & 5 & 7 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}.$$

The rank of the last matrix is equal to 2 since, for instance, $\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \neq 0$. Consequently, the rank of the given matrix is equal to 2 as well.

431. Determine the rank of the matrix $\begin{pmatrix} 4 & 3 & 2 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{pmatrix}$.

Solution. Subtract the elements of the 3rd column from the elements of the 4th column:

$$\begin{pmatrix} 4 & 3 & 2 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 4 & 3 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}.$$

Delete the fourth column:

$$\begin{pmatrix} 4 & 3 & 2 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 4 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

Since $\begin{vmatrix} 4 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{vmatrix} = 24 \neq 0$, the rank of the matrix is equal to 3.

432. Determine the rank and find the base minors of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 2 & 0 & 4 & 0 & 0 \end{pmatrix} .$$

Solution. We have

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 2 & 0 & 4 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 4 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}; \ r(A) = 2.$$

The base minors are the second-order nonzero minors of the matrix:

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 4 & 0 \end{vmatrix}.$$

Thus the matrix A possesses 8 base minors.

433. How many second-order minors does the following matrix have:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}?$$

Write out all these minors.

Solution. The matrix possesses $C_3^2 \cdot C_3^2 = 3 \cdot 3 = 9$ second-order minors:

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix},$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}, \quad \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

434. Determine the rank of the matrix $A = \begin{pmatrix} \lambda & 5\lambda & -\lambda \\ 2\lambda & \lambda & 10\lambda \\ -\lambda & -2\lambda & -3\lambda \end{pmatrix}$.

435. Determine the rank of the matrix $\begin{pmatrix} 1 & 2 & 3 & 6 \\ 2 & 3 & 1 & 6 \\ 3 & 1 & 2 & 6 \end{pmatrix}$.

436. Determine the rank of the matrix $A = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 3 & 0 \end{pmatrix}$ and find its base minors.

437. Determine the rank of the matrix $A = \begin{pmatrix} 1 & 2 & 1 & 3 & 4 \\ 3 & 4 & 2 & 6 & 8 \\ 1 & 2 & 1 & 3 & 4 \end{pmatrix}$ and find its base minors.

4.5. Investigation of a System of *m* Linear Equations in *n* Unknowns

Given the system of m linear equations in n unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m. \end{cases} \qquad (1)$$

The *solution* of this system is a set of n numbers $(\bar{x}_1; \bar{x}_2; \dots; \bar{x}_n)$ which, being substituted for the unknowns in the equations, turn the equations into identities. The system of equations is called *consistent* if it possesses at least one solution $(\bar{x}_1; \bar{x}_2; \dots; \bar{x}_n)$. Now if the system does not possess a single solution it is said to be *inconsistent*.

A consistent system is said to be *determinate* if it possesses only one solution and *indeterminate* if it has more than one solution.

The matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \dots a_{1n} \\ a_{21} & a_{22} \dots a_{2n} \\ \dots & \dots \\ a_{m1} & a_{m2} \dots a_{mn} \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} a_{11} & a_{12} \dots a_{1n} & b_1 \\ a_{21} & a_{22} \dots a_{2n} & b_2 \\ \dots & \dots & \dots \\ a_{m1} & a_{m2} \dots a_{mn} & b_m \end{pmatrix}$$

are called a *matrix* and an *augmented matrix of system* (1) respectively.

For system (1) *to be consistent, it is necessary and sufficient that the rank of the matrix of the system be equal to the rank of its augmented matrix* (**Kronecker-**

Capelli's Theorem). Thus, system (1) is consistent if and only if $r(A) = r(A_1) = r$. In this case the number r is called the *rank of system* (1).

If $b_1 = b_2 = \ldots = b_m = 0$, the system of linear equations (1) is called *homogeneous*. A homogeneous system of equations is always consistent.

If the rank of a consistent system is equal to the number of unknowns (i.e. $r = n$), the system is determinate.

Now if the rank of a consistent system is smaller than the number of the unknowns, the system is indeterminate. Let us take this last case, that is suppose that system (1) is consistent, with $r < n$. Let us consider some base minor of the matrix A and isolate in this minor an arbitrary row. The elements of this row are the coefficients in r unknowns in one of the equations of system (1). We shall call these r unknowns the *base unknowns* of the system of equations under consideration. The rest $n - r$ unknowns of system (1) will be called *free unknowns*.

Let us isolate from system (1) a system of r equations among whose coefficients there are elements of the base minor. The base unknowns in the isolated system will be left in the left-hand parts of the equations and the terms containing free unknowns will be transposed into the right-hand parts. In the system of equations obtained let us express the base unknowns in terms of the free unknowns (using Cramer's rule, for instance).

In this way, assigning arbitrary values to the free unknowns, we can find the corresponding values of the base unknowns. Consequently (it was mentioned above), system (1) possesses infinitely many solutions.

438. Investigate the system of equations

$$\begin{cases} x_1 + 3x_2 + 5x_3 + 7x_4 + 9x_5 = 1, \\ x_1 - 2x_2 + 3x_3 - 4x_4 + 5x_5 = 2, \\ 2x_1 + 11x_2 + 12x_3 + 25x_4 + 22x_5 = 4. \end{cases}$$

Solution. Determine the ranks of the matrix and of the augmented matrix of the system. Write out the augmented matrix

$$A_1 = \left(\begin{array}{ccccc|c} 1 & 3 & 5 & 7 & 9 & 1 \\ 1 & -2 & 3 & -4 & 5 & 2 \\ 2 & 11 & 12 & 25 & 22 & 4 \end{array}\right).$$

The vertical line isolates the elements of the matrix of the system (matrix A) from the constant terms.

Add to the elements of the 2nd row the corresponding elements of the 3rd row:

$$A_1 \sim \left(\begin{array}{ccccc|c} 1 & 3 & 5 & 7 & 9 & 1 \\ 3 & 9 & 15 & 21 & 27 & 6 \\ 2 & 11 & 12 & 25 & 22 & 4 \end{array}\right).$$

Divide all the elements of the 2nd row by 3:

$$A_1 \sim \left(\begin{array}{ccccc|c} 1 & 3 & 5 & 7 & 9 & 1 \\ 1 & 3 & 5 & 7 & 9 & 2 \\ 2 & 11 & 12 & 25 & 22 & 4 \end{array}\right).$$

Subtract from the elements of the 2nd row the corresponding elements of the 1st row:

$$A_1 \sim \left(\begin{array}{ccccc|c} 1 & 3 & 5 & 7 & 9 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 11 & 12 & 25 & 22 & 4 \end{array}\right);$$

$$A \sim \begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 11 & 12 & 25 & 22 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 2 & 11 & 12 & 25 & 22 \end{pmatrix}.$$

It is easy to see that $r(A) = 2$, $r(A_1) = 3$, i.e. $r(A) \neq r(A_1)$; consequently, the system is inconsistent.

439. Investigate the system of equations

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 14, \\ 3x_1 + 2x_2 + x_3 = 10, \\ x_1 + x_2 + x_3 = 6, \\ 2x_1 + 3x_2 - x_3 = 5, \\ x_1 + x_2 = 3. \end{cases}$$

Solution. The augmented matrix of the system has the form

$$A_1 = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 3 & 2 & 1 & 10 \\ 1 & 1 & 1 & 6 \\ 2 & 3 & -1 & 5 \\ 1 & 1 & 0 & 3 \end{array}\right).$$

Add the elements of the 2nd row to the corresponding elements of the 1st and the 4th rows:

$$A_1 \sim \left(\begin{array}{ccc|c} 4 & 4 & 4 & 24 \\ 3 & 2 & 1 & 10 \\ 1 & 1 & 1 & 6 \\ 5 & 5 & 0 & 15 \\ 1 & 1 & 0 & 3 \end{array}\right).$$

Divide the elements of the 1st row by 4 and the elements of the 4th row by 5:

$$A_1 \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & 2 & 1 & 10 \\ 1 & 1 & 1 & 6 \\ 1 & 1 & 0 & 3 \\ 1 & 1 & 0 & 3 \end{array}\right).$$

Subtract from the elements of the 3rd row the corresponding elements of the 1st

row and from the elements of the 5th row the elements of the 4th row:

$$A_1 \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & 2 & 1 & 10 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Delete the 3rd and the 5th rows:

$$A_1 \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & 2 & 1 & 10 \\ 1 & 1 & 0 & 3 \end{array}\right); \qquad A \sim \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Find the determinant of the last matrix:

$$\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1 \neq 0.$$

Consequently, $r(A) = 3$. The rank of the augmented matrix is also 3 since the determinant we have found is a minor of the matrix A_1.

Thus we see that the system is consistent. To solve it, take the first, the third and the fifth equation, for example:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 14, \\ x_1 + x_2 + x_3 = 6, \\ x_1 + x_2 = 3. \end{cases}$$

From this we easily find that $x_1 = 1, x_2 = 2, x_3 = 3$.

440. Investigate the system of equations

$$\begin{cases} x_1 + 5x_2 + 4x_3 + 3x_4 = 1, \\ 2x_1 - x_2 + 2x_3 - x_4 = 0, \\ 5x_1 + 3x_2 + 8x_3 + x_4 = 1. \end{cases}$$

Solution. We have

$$A_1 = \left(\begin{array}{cccc|c} 1 & 5 & 4 & 3 & 1 \\ 2 & -1 & 2 & -1 & 0 \\ 5 & 3 & 8 & 1 & 1 \end{array}\right).$$

Subtract the 1st row from the 3rd row:

$$A_1 \sim \left(\begin{array}{cccc|c} 1 & 5 & 4 & 3 & 1 \\ 2 & -1 & 2 & -1 & 0 \\ 4 & -2 & 4 & -2 & 0 \end{array}\right).$$

Divide the elements of the 3rd row by 2 and subtract the 2nd row from the resulting 3rd row:

$$A_1 \sim \left(\begin{array}{cccc|c} 1 & 5 & 4 & 3 & 1 \\ 2 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Delete the 3rd row:

$$A_1 \sim \left(\begin{array}{cccc|c} 1 & 5 & 4 & 3 & 1 \\ 2 & -1 & 2 & -1 & 0 \end{array}\right).$$

It is easy to see that $r(A) = r(A_1) = 2$. Consequently, the system is consistent.

Take the first and the second equation of the given system:

$$\begin{cases} x_1 + 5x_2 + 4x_3 + 3x_4 = 1, \\ 2x_1 - x_2 + 2x_3 - x_4 = 0. \end{cases}$$

Take x_1 and x_2 as the base unknowns. This can be done since the determinant $\begin{vmatrix} 1 & 5 \\ 2 & -1 \end{vmatrix}$ from the coefficients in these unknowns is nonzero. The free unknowns are x_3 and x_4. Having rewritten the system in the form

$$\begin{cases} x_1 + 5x_2 = 1 - 4x_3 - 3x_4, \\ 2x_1 - x_2 = -2x_3 + x_4, \end{cases}$$

express x_1 and x_2 in terms of x_3 and x_4:

$$x_1 = \frac{\begin{vmatrix} 1 - 4x_3 - 3x_4 & 5 \\ -2x_3 + x_4 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 5 \\ 2 & -1 \end{vmatrix}} = -\frac{6}{11}x_3 - \frac{8}{11}x_4 - \frac{1}{11},$$

$$x_2 = \frac{\begin{vmatrix} 1 & 1 - 4x_3 - 3x_4 \\ 2 & -2x_3 + x_4 \end{vmatrix}}{-11} = -\frac{6}{11}x_3 + \frac{7}{11}x_4 + \frac{2}{11}.$$

Putting $x_3 = u$, $x_4 = v$, we obtain the solution of the system in the form

$$x_1 = -\frac{6}{11}u + \frac{8}{11}v - \frac{1}{11}, \quad x_2 = -\frac{6}{11}u + \frac{7}{11}v + \frac{2}{11},$$

$$x_3 = u, \quad x_4 = v.$$

Assigning to u and v different numerical values, we obtain different solutions of the given system of equations.

Investigate the following systems of equations:

441. $$\begin{cases} 3x_1 + 2x_2 = 4, \\ x_1 - 4x_2 = -1, \\ 7x_1 + 10x_2 = 12, \\ 5x_1 + 6x_2 = 8, \\ 3x_1 - 16x_2 = -5. \end{cases}$$

442. $$\begin{cases} x_1 + 5x_2 + 4x_3 = 1, \\ 2x_1 + 10x_2 + 8x_3 = 3, \\ 3x_1 + 15x_2 + 12x_3 = 5. \end{cases}$$

443. $$\begin{cases} x_1 - 3x_2 + 2x_3 = -1, \\ x_1 + 9x_2 + 6x_3 = 3, \\ x_1 + 3x_2 + 4x_3 = 1. \end{cases}$$

4.6. Gauss' Method of Solution of a System of Linear Equations

A numerical solution of algebraic linear equations by means of determinants is convenient for systems of two and three equations. In the case of systems of more than three equations it is advantageous to use *Gauss' method* which consists in consecutive elimination of unknowns (*Gaussian elimination*). We shall elucidate the meaning of the method by considering a system of four equations in four unknowns:

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z + a_{14}u = a_{15}, & (a) \\ a_{21}x + a_{22}y + a_{23}z + a_{24}u = a_{25}, & (b) \\ a_{31}x + a_{32}y + a_{33}z + a_{34}u = a_{35}, & (c) \\ a_{41}x + a_{42}y + a_{43}z + a_{44}u = a_{45}. & (d) \end{cases}$$

Asume that $a_{11} \neq 0$ (if $a_{11} = 0$ we shall change the order in which the equations follow putting in the first line the equation in which the coefficient in x is nonzero).

1st step: divide equation (a) by a_{11}, multiply the resulting equation by a_{21} and subtract it from (b); then multiply by a_{31} and subtract from (c); finally, multiply by a_{41} and subtract from (d). The first step yields the system

$$\begin{cases} x + b_{12}y + b_{13}z + b_{14}u = b_{15}, & (e) \\ \qquad b_{22}y + b_{23}z + b_{24}u = b_{25}, & (f) \\ \qquad b_{32}y + b_{33}z + b_{34}u = b_{35}, & (g) \\ \qquad b_{42}y + b_{43}z + b_{44}u = b_{45}, & (h) \end{cases}$$

b_{ij} being obtained from a_{ij} by the following formulas:

$$b_{1j} = a_{1j}/a_{11} \quad (j = 2, 3, 4, 5);$$
$$b_{ij} = a_{ij} - a_{i1}b_{1j} \quad (i = 2, 3, 4; j = 2, 3, 4, 5).$$

2nd step: perform the same actions on equations (f), (g), (h) as on equations (a), (b), (c), (d), etc. As a result, the original system acquires the so-called stepped form:

$$\begin{cases} x + b_{12}y + b_{13}z + b_{14}u = b_{15}, \\ \qquad y + c_{23}z + c_{24}u = c_{25}, \\ \qquad\qquad z + d_{34}u = d_{35}, \\ \qquad\qquad\qquad u = e_{45}. \end{cases}$$

The transformed system makes it possible to obtain all the unknowns in a consecutive order without difficulty.

444. Solve the following system of equations:

$$\begin{aligned} 36.47x + 5.28y + 6.34z &= 12.26, &(a)\\ 7.33x + 28.74y + 5.86z &= 15.15, &(b)\\ 4.63x + 6.31y + 26.17z &= 25.22. &(c)\end{aligned}$$

Solution. Dividing equation (*a*) by 36.47, we obtain

$$x + 0.1447y + 0.1738z = 0.3361. \qquad (*)$$

Multiplying equation (*) by 7.33 and subtracting the result from (b), we get

$$27.6793y + 4.586z = 12.6864;$$

now we multiply equation (*) by 4.63 and subtract the result from (c); we get

$$5.64y + 25.3653z = 23.6639.$$

Thus we arrive at a system of equations

$$\begin{cases} 27.6793y + 4.586z = 12.6864, & (d)\\ 5.64y + 25.3653z = 23.6639. & (e)\end{cases}$$

Dividing equation (d) by 27.68, we obtain

$$y + 0.1657z = 0.4583. \qquad (**)$$

Multiplying equation (**) by 5.64 and subtracting from (e), we get $24.4308z = 21.0791$. Consequently, $z = 0.8628$. Then

$$y = 0.4583 - 0.1657 \cdot 0.8628 = 0.3153,$$
$$x = 0.3361 - 0.1447 \cdot 0.3153 = 0.1738 \cdot 0.8628 = 0.1405.$$

Thus we have $x = 0.1405$, $y = 0.3153$, $z = 0.8628$.

In practical calculations it is more convenient to reduce to the stepped form not the system of equations itself but a matrix composed of the coefficients in the

unknowns and constant terms:

$$\left(\begin{array}{ccc|c} 36.47 & 5.28 & 6.34 & 12.26 \\ 7.33 & 28.74 & 5.86 & 15.15 \\ 4.63 & 6.31 & 28.17 & 25.22 \end{array}\right).$$

Now we introduce the fifth, so-called *check column* whose each element is the sum of four elements of the given row:

$$\left(\begin{array}{ccc|c|c} 36.47 & 5.28 & 6.34 & 12.26 & 60.35 \\ 7.33 & 28.74 & 5.86 & 15.15 & 57.08 \\ 4.63 & 6.31 & 26.17 & 25.22 & 62.33 \end{array}\right).$$

When the elements of a matrix undergo a linear transformation, the same transformation must be used for the elements of the check column. It is easy to see that each element of the check column of the transformed matrix is equal to the sum of the elements of the corresponding row. We shall denote the transition from one matrix to another by means of the equivalence sign:

$$\left(\begin{array}{ccc|c|c} 36.47 & 5.28 & 6.34 & 12.26 & 60.35 \\ 7.33 & 28.74 & 5.86 & 15.15 & 57.08 \\ 4.63 & 6.31 & 26.17 & 25.22 & 62.33 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c|c} 1 & 0.1447 & 0.1738 & 0.3361 & 1.6547 \\ 7.33 & 28.74 & 5.86 & 15.15 & 57.08 \\ 4.63 & 6.31 & 26.17 & 25.22 & 62.33 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c|c} 1 & 0.1447 & 0.1738 & 0.3361 & 1.6547 \\ 0 & 27.6793 & 4.586 & 12.6864 & 44.9516 \\ 0 & 5.64 & 25.3653 & 23.6639 & 54.6688 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c|c} 1 & 0.1447 & 0.1738 & 0.3361 & 1.6547 \\ 0 & 1 & 0.1657 & 0.4583 & 1.6240 \\ 0 & 5.64 & 25.3653 & 23.6639 & 54.6688 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c|c} 1 & 0.1447 & 0.1738 & 0.3361 & 1.6547 \\ 0 & 1 & 0.1657 & 0.4583 & 1.6240 \\ 0 & 0 & 24.4308 & 21.0791 & 45.5094 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c|c} 1 & 0.1447 & 0.1738 & 0.3361 & 1.6547 \\ 0 & 1 & 0.1657 & 0.4583 & 1.6240 \\ 0 & 0 & 1 & 0.8628 & 1.8629 \end{array}\right).$$

Proceeding from the matrix obtained, we write out the transformed system and find the solution:

$$\begin{aligned} z &= 0.8628, \\ y &= 0.4583 - 0.1657 \cdot 0.8628 = 0.3153, \\ x &= 0.3361 - 0.1738 \cdot 0.8628 - 0.1447 \cdot 0.3153 = 0.1405. \end{aligned}$$

If the system possesses a unique solution, the stepped system of equations can be reduced to a triangular form, that is, a system whose last equation contains one

unknown. In the case of an undetermined system, that is a system in which the number of unknowns is larger than the number of linearly independent equations, due to which fact the system admits an infinite number of solutions, a triangular system cannot be obtained since the last equation contains more than one unknown.

Now if the system of equations is inconsistent, then, having been reduced to a stepped form, it contains at least one equation of the form $0 = 1$, that is, an equation in which all the unknowns have zero coefficients and the right-hand side is nonzero. Such a system has no solution.

445. Solve the following system of equations:

$$\begin{cases} 3x + 2y + z = 5, \\ x + y - z = 0, \\ 4x - y + 5z = 3. \end{cases}$$

Solution. Transform the matrix into an equivalent one:

$$\left(\begin{array}{ccc|c|c} 3 & 2 & 1 & 5 & 11 \\ 1 & 1 & -1 & 0 & 1 \\ 4 & -1 & 5 & 3 & 11 \end{array}\right) \sim \left(\begin{array}{ccc|c|c} 1 & 1 & -1 & 0 & 1 \\ 3 & 2 & 1 & 5 & 11 \\ 4 & -1 & 5 & 3 & 11 \end{array}\right)$$

(to simplify the calculations, we have interchanged the positions of the first and the second equation).

Subtract the 1st row multiplied by 3 and by 4 from the rest two rows

$$\left(\begin{array}{ccc|c|c} 1 & 1 & -1 & 0 & 1 \\ 0 & -1 & 4 & 5 & 8 \\ 0 & -5 & 9 & 3 & 7 \end{array}\right).$$

Having changed signs in the 2nd row and multiplied it by 5, we add the result of the 3rd row:

$$\left(\begin{array}{cc|c|c|c} 1 & 1 & -1 & 0 & 1 \\ 0 & 1 & -4 & -5 & -8 \\ 0 & 0 & -11 & -22 & -33 \end{array}\right) \sim \left(\begin{array}{cc|c|cc} 1 & 1 & -1 & 0 & 1 \\ 0 & 1 & -4 & -5 & -8 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right)$$

(we have divided the last row by -11).

The system of equations have reduced to a triangular form:

$$\begin{cases} x + y - z = 0, \\ \quad y - 4z = -5, \\ \qquad z = 2. \end{cases}$$

It possesses a unique solution. The last equation yields $z = 2$; substituting this value into the second equation, we obtain $y = 3$ and, finally, we find from the first equation $x = -1$.

Solve the following system of equations:

446. $\begin{cases} 2x_1 + x_2 - x_3 = 5, \\ x_1 - 2x_2 + 3x_3 = -3, \\ 7x_1 + x_2 - x_3 = 10. \end{cases}$

447. $\begin{cases} x_1 + x_2 - x_3 + x_4 = 4, \\ 2x_1 - x_2 + 3x_3 - 2x_4 = 1, \\ x_1 - x_3 + 2x_4 = 6, \\ 3x_1 - x_2 + x_3 - x_4 = 0. \end{cases}$

448. $\begin{cases} 3x_1 - x_2 + x_3 + 2x_5 = 18, \\ 2x_1 - 5x_2 + x_4 + x_5 = -7, \\ x_1 - x_4 + 2x_5 = 8, \\ 2x_2 + x_3 + x_4 - x_5 = 10, \\ x_1 + x_2 - 3x_3 + x_4 = 1. \end{cases}$

449. $\begin{cases} 4x_1 + 2x_2 + 3x_3 = -2, \\ 2x_1 + 8x_2 - x_3 = 8, \\ 9x_1 + x_2 + 8x_3 = 0. \end{cases}$

450. $\begin{cases} 0.04x - 0.08y + 4z = 20, \\ 4x + 0.24y - 0.08z = 8, \\ 0.09x + 3y - 0.15z = 9. \end{cases}$

451. $\begin{cases} 3.21x + 0.71y + 0.34z = 6.12, \\ 0.43x + 4.11y + 0.22z = 5.71, \\ 0.17x + 0.16y + 4.73z = 7.06. \end{cases}$

4.7. Application of Jordan-Gauss' Method to the Solution of Systems of Linear Equations

When solving systems of linear equations by Gauss' method, we considered the matrix method with a check column as a result of which the given system reduced to a triangular form (see solution of problem 444). For what follows it is necessary to get acquainted with the *modified method of Jordan-Gauss* making it possible to find directly the values of the unknowns.

Suppose we are given a system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m. \end{cases} \tag{1}$$

We shall choose in the matrix A of this system a nonzero element a_{qp}. This element is called a *resolvent element*, the pth column of the matrix A is called a *resolvent* column, and the qth row is called a *resolvent row*.

Let us consider a new system of equations

$$\begin{cases} a'_{11}x_1 + a'_{12}x_2 + \ldots + a'_{1n}x_n = b'_1, \\ a'_{21}x_1 + a'_{22}x_2 + \ldots + a'_{2n}x_n = b'_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a'_{m1}x_1 + a'_{m2}x_2 + \ldots + a'_{mn}x_n = b'_m \end{cases} \qquad (2)$$

with the matrix A' whose coefficients and constant terms are found from the formulas

$$\left.\begin{aligned} a'_{ij} &= a_{ij} - \frac{a_{ip} \cdot a_{qi}}{a_{qp}} \\ b'_i &= b_i - \frac{a_{ip} \cdot b_q}{a_{qp}} \end{aligned}\right\}, \quad \text{if} \quad i \neq q.$$

In particular, $a_{ip} = 0$ if $i \neq q$. Now if $i = q$, we assume $a'_{qj} = a_{qj}$, $b'_q = b_q$. Thus the qth equations in systems (1) and (2) are the same and the coefficients in x_p are equal to zero in all the equations of system (2) except for the qth.

It should be remembered that systems (1) and (2) are simultaneously consistent or inconsistent. If they are consistent they are also identical (their solutions coincide).

To determine the element a'_{ij} of the matrix A', it is useful to recall the so-called "rule of rectangle".

Let us consider four elements of the matrix A: a_{ij} (the element to be transformed), a_{qp} (resolvent element) and the elements a_{ip} and a_{qj}. To determine the element a'_i, we must subtract from the element a_{ij} the product of the elements a_{ip} and a_{qj}, located at the opposite vertices of a rectangle, divided by the resolvent element a_{qp}:

$$\begin{array}{ccc} a_{ij} & \cdots\cdots\cdots & a_{ip} \\ \vdots & & \vdots \\ a_{qj} & \cdots\cdots\cdots & a_{qp} \end{array}$$

We can transform system (2) in a similar way, taking the element $a'_{sr} \neq 0$ as the resolvent element of the matrix A', with $s \neq q$, $r \neq p$. As a result of this transformation, all the coefficients in x_r, except for a_{sr}, will vanish. The system that will be obtained can be transformed again and so long. If $r = n$ (the rank of the system is equal to the number of unknowns), a series of transformations will yield a system of equations of the form

$$\begin{aligned} k_1x_1 &= l_1, \\ k_2x_2 &= l_2, \\ &\ldots\ldots \\ k_nx_n &= l_n, \end{aligned}$$

which can be used to find the values of the unknowns. This method of solution based on a consecutive elimination of the unknowns is known as the method of Jordan-Gauss.

452. Given the matrix of the system of linear equations

$$A = \begin{pmatrix} 5 & 4 & 6 & -1 & 7 \\ 8 & 1 & 3 & 2 & 0 \\ 0 & 1 & 5 & 3 & -1 \\ 7 & -6 & 5 & -4 & 3 \end{pmatrix}.$$

When solving this system by the Jordan-Gauss method, $a_{23} = 3$ was taken as the resolvent element. Find the elements a'_{24}, a'_{13}, a'_{44} of the transformed matrix.

Solution. Since a_{24} is an element of the resolvent row, we have $a'_{24} = a_{24} = 2$. The element a_{13} belongs to the resolvent column, therefore $a'_{13} = 0$. The element a'_{44} can be found by the rule of rectangle:

$$A = \begin{pmatrix} 5 & 4 & 6 & -1 & 7 \\ 8 & 1 & \boxed{3} & 2 & 0 \\ 0 & 1 & 5 & 3 & -1 \\ 7 & -6 & 5 & -4 & 3 \end{pmatrix}$$

$$a'_{44} = a_{44} - \frac{a_{24} \cdot a_{43}}{a_{23}} = -4 - \frac{2 \cdot 5}{3} = -7\frac{1}{3}.$$

453. Solve the following system of equations:

$$\begin{cases} x_1 + x_2 - 3x_3 + 2x_4 = 6, \\ x_1 - 2x_2 - x_4 = -6, \\ x_2 + x_3 + 3x_4 = 16, \\ 2x_1 - 3x_2 + 2x_3 = 6. \end{cases}$$

Solution. Let us tabulate the coefficients, constant terms and the sums of the coefficients and constant terms (Σ is a check column):

x_1	x_2	x_3	x_4	b	Σ
$\boxed{1}$	1	−3	2	6	7
1	−2	0	−1	−6	−8
0	1	1	3		21
2	−3	2	0	6	7

We have taken the coefficient in x_1 in the first equation as the resolvent element. Let us replace by zeros all the elements of the 1st column, except for the resolvent element, leaving unchanged the row containing the resolvent element (the resolvent row):

x_1	x_2	x_3	x_4	b	Σ
1	1	-3	2	6	7
0					
0					
0					

Applying the rule of rectangle, we shall fill in the empty cells of the table (the same rule will be applied to the Σ column):

x_1	x_2	x_3	x_4	b	Σ
1	1	-3	2	6	7
0	-3	3	-3	-12	-15
0	1	1	3	16	21
0	-5	8	-4	-6	-7

Pay attention to the fact that the check column contains the sums of the elements of the respective rows. Dividing the elements of the 2nd row by -3, we obtain the following table:

x_1	x_2	x_3	x_4	b	Σ
1	1	−3	2	6	7
0	$\boxed{1}$	−1	1	4	5
0	1	1	3	16	21
0	−5	8	−4	−6	−7

Let us accept the 2nd element of the 2nd row as the resolvent element, rewrite the 1st column as it is, replace by zeros the elements of the 2nd column, except for the resolvent element, leave unchanged the 2nd (resolvent) row, and transform by the rule of rectangle the elements of the remaining cells of the table:

x_1	x_2	x_3	x_4	b	Σ
1	0	−2	1	2	2
0	1	−1	1	4	5
0	0	2	2	12	16
0	0	3	1	14	18

Dividing the elements of the 3rd row by 2, we obtain

x_1	x_2	x_3	x_4	b	Σ
1	0	-2	1	2	2
0	1	-1	1	4	5
0	0	$\boxed{1}$	1	6	8
0	0	3	1	14	18

Next we transform the table taking the 3rd element of the 3rd column as the resolvent element:

x_1	x_2	x_3	x_4	b	Σ
1	0	0	3	1	18
0	1	0	2	10	13
0	0	1	1	6	8
0	0	0	-2	-4	-6

We divide the elements of the 4th row by -2:

x_1	x_2	x_3	x_4	b	Σ
1	0	0	3	14	18
0	1	0	2	10	13
0	0	1	1	6	8
0	0	0	$\boxed{1}$	2	3

Now we transform the table taking the 4th element of the 4th row as the resolvent element:

x_1	x_2	x_3	x_4	b	Σ
1	0	0	0	8	9
0	1	0	0	6	7
0	0	1	0	4	5
0	0	0	1	2	3

As a result of these operations we obtain the following system of equations:

$$\begin{cases} 1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 8, \\ 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 6, \\ 0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 = 4, \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 2, \end{cases}$$

i.e. $x_1 = 8$, $x_2 = 6$, $x_3 = 4$, $x_4 = 2$.

454. Solve the following system of equations:

$$\begin{cases} x_1 + x_2 - 2x_3 + x_4 = 1, \\ x_1 - 3x_2 + x_3 + x_4 = 0, \\ 4x_1 - x_2 - x_3 - x_4 = 1, \\ 4x_1 + 3x_2 - 4x_3 - x_4 = 2. \end{cases}$$

Solution. Compile a table:

$\boxed{1}$	1	−2	1	1	2
1	−3	1	1	0	0
4	−1	−1	−1	1	2
4	3	−4	−1	2	4

The 1st element of the 1st column is the resolvent element:

1	1	−2	1	1	2
0	−4	3	0	−1	−2
0	−5	7	−5	−3	−6
0	−1	4	−5	−2	−4

Change signs in the 4th row:

1	1	−2	1	1	2
0	−4	3	0	−1	−2
0	−5	7	−5	−3	−6
0	$\boxed{1}$	−4	5	2	4

The 4th element of the 2nd column is the resolvent element:

1	0	2	−4	−1	−2
0	0	−13	20	7	14
0	0	−13	20	7	14
0	1	−4	5	2	4

Subtract the 2nd row from the 3rd:

1	0	2	−4	−1	−2
0	0	−13	20	7	14
0	0	0	0	0	0
0	1	−4	5	2	4

The 3rd row may be deleted:

1	0	2	−4	−1	−2
0	0	−13	$\boxed{20}$	7	14
0	1	−4	5	2	4

The 4th element of the 2nd row is the resolvent element:

1	0	−0.6	0	0.4	0.8
0	0	−13	20	7	14
0	1	−0.75	0	0.25	0.5

The rank of the matrix is 3; consequently, the system contains three base unknowns x_1, x_2 and x_4 and one free unknown x_3. We obtain a system of equations

$$\begin{cases} 1 \cdot x_1 + 0 \cdot x_2 - 0.6x_3 + 0 \cdot x_4 = 0.4, \\ 0 \cdot x_1 + 0 \cdot x_2 - 13x_3 + 20x_4 = 7, \\ 0 \cdot x_1 + 1 \cdot x_2 - 0.75x_3 + 0 \cdot x_4 = 0.25. \end{cases}$$

From this we have

$$x_1 = 0.4 + 0.6x_3, \quad x_2 = 0.25 + 0.75x_3, \quad x_4 = 0.35 + 0.65x_3.$$

Thus, the solution of the system has the form

$$x_1 = 0.4 + 0.6u, \quad x_2 = 0.25 + 0.75u, \quad x_3 = u, \quad x_4 = 0.35 + 0.65u,$$

where u is an arbitrary number.

455. Solve the following system of equations:

$$\begin{cases} 6x - 5y + 7z + 8t = 3, \\ 3x + 11y + 2z + 4t = 6, \\ 3x + 2y + 3z + 4t = 1, \\ x + y + z = 0, \end{cases}$$

Solution. Compile a table:

6	−5	7	8	3	19
3	11	2	4	6	26
3	2	3	4	1	13
$\boxed{1}$	1	1	0	0	3

The 4th element of the 1st column is the resolvent element:

0	−11	$\boxed{1}$	8	3	1
0	8	−1	4	6	17
0	−1	0	4	1	4
1	1	1	0	0	3

The 1st element of the 3rd column is the resolvent element:

0	−11	1	8	3	1
0	−3	0	12	9	18
0	−1	0	4	1	4
1	12	0	−8	−3	2

Change signs of the elements of the 3rd row:

0	−11	1	8	3	1
0	−3	0	12	9	18
0	$\boxed{1}$	0	−4	−1	−4
1	12	0	−8	−3	2

The 3rd element of the second column is the resolvent element:

0	0	1	−36	−8	−43
0	0	0	0	6	6
0	1	0	−4	−1	−4
1	0	0	40	9	50

As a result we arrive at a system

$$\begin{cases} 0 \cdot x + 0 \cdot y + 1 \cdot z - 36t = -8, \\ 0 \cdot x + 0 \cdot y + 0 \cdot z + 0 \cdot t = 6, \\ 0 \cdot x + 1 \cdot y + 0 \cdot z - 4t = -1 \\ 1 \cdot x + 0 \cdot y + 0 \cdot z + 40t = 9. \end{cases}$$

It is easy to see that the second equation is not satisfied by any values of x, y, z and t. Thus, the system of equations we have obtained and the given system are inconsistent.

456. Apply the Jordan-Gauss method to determine the rank of the matrix

$$A = \begin{pmatrix} 7 & -1 & 3 & 5 \\ 1 & 3 & 5 & 7 \\ 4 & 1 & 4 & 6 \\ 3 & -2 & -1 & -1 \end{pmatrix}.$$

Solution. Compile a table:

7	-1	3	5	14
$\boxed{1}$	3	5	7	16
4	1	4	6	15
3	-2	-1	-1	-1

The last (check) column contains the sums of the elements of the corresponding rows. The 2nd element of the 1st column is the resolvent element:

0	-22	-32	-44	-98
1	3	5	7	16
0	-11	-16	-22	-49
0	-11	-16	-22	-49

Divide the elements of the 1st row by -2:

0	11	16	22	49
1	3	5	7	16
0	11	16	22	49
0	11	16	22	49

Subtract the elements of the first row from the respective elements of the 4th and the 3rd row:

0	11	16	22	49
1	3	5	7	16
0	0	0	0	0
0	0	0	0	0

Having deleted the 3rd and the 4th row, we obtain a table:

0	11	16	22	49
1	3	5	7	16

Any second-order determinant of the matrix we have obtained is nonzero. Consequently, $r(A) = 2$.

Use the Jordan-Gauss method to solve the following systems of equations:

457. $$\begin{cases} x_1 + 2x_2 + x_3 = 8, \\ x_2 + 3x_3 + x_4 = 15, \\ 4x_1 + x_3 + x_4 = 11, \\ x_1 + x_2 + 5x_4 = 23. \end{cases}$$

458. $$\begin{cases} x_1 - x_2 + x_3 - x_4 = -2, \\ x_1 + 2x_2 - 2x_3 - x_4 = -5, \\ 2x_1 - x_2 - 3x_3 + 2x_4 = -1, \\ x_1 + 2x_2 + 3x_3 - 6x_4 = -10. \end{cases}$$

459. $$\begin{cases} x_1 + 5x_2 - 2x_3 - 3x_4 = 1, \\ 7x_1 + 2x_2 - 3x_3 - 4x_4 = 2, \\ x_1 + x_2 + x_3 + x_4 = 5, \\ 2x_1 + 3x_2 + 2x_3 - 3x_4 = 4, \\ x_1 - x_2 - x_3 - x_4 = -2. \end{cases}$$

460. Determine the rank of the following matrix using the Jordan-Gauss method:

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 1 & 3 \\ 3 & 1 & 2 & 5 \\ 2 & 2 & 2 & 3 \end{pmatrix}.$$

Chapter 5

Fundamentals of Linear Algebra

5.1. Linear Spaces

5.1.1. Basic notions. Let us consider a set R of the elements $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$, such that the sum $\mathbf{x} + \mathbf{y} \in R$ is defined for any two elements $\mathbf{x} \in R$ and $\mathbf{y} \in R$ and the product $\lambda\mathbf{x} \in R$ is defined for any element $\mathbf{x} \in R$ and any real number λ.

If the summation of the members of the set R and multiplication of a member of that set by a real number satisfy the conditions:

1°. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$;

2°. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$;

3°. there exists an element $\mathbf{0} \in R$ (null-element) such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for any $\mathbf{x} \in R$;

4°. there exists an element $\mathbf{y} \in R$ for every element $\mathbf{x} \in R$ such that $\mathbf{x} + \mathbf{y} = \mathbf{0}$ (in what follows we shall write $\mathbf{y} = -\mathbf{x}$, i.e. $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$);

5°. $1 \cdot \mathbf{x} = \mathbf{x}$;

6°. $\lambda(\mu\mathbf{x}) = (\lambda\mu)\mathbf{x}$;

7°. $(\lambda + \mu)\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x}$;

8°. $\lambda(\mathbf{x} + \mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}$,

then the set R is said to be a *linear* (or *vector*) *space*, and the elements $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ of that space are called *vectors*.

For instance, the set of all geometric vectors is a linear space since the operations of addition and multiplication by a number complying with the conditions formulated above are defined for the elements of that set.

The *difference* of two vectors $\mathbf{x}$ and $\mathbf{y}$ of a linear space is a vector $\mathbf{v}$ of that space such that $\mathbf{y} + \mathbf{v} = \mathbf{x}$. The difference of the vectors $\mathbf{x}$ and $\mathbf{y}$ is designated as $\mathbf{x} - \mathbf{y}$, i.e. $\mathbf{x} - \mathbf{y} = \mathbf{v}$. It is easy to prove that $\mathbf{x} - \mathbf{y} = \mathbf{x} + (-\mathbf{y})$.

The following theorems hold true:

1. *Every linear space contains only one null-element.*
2. *Each element of a linear space is associated with only one opposite element.*
3. *The equality* $\mathbf{0} \cdot \mathbf{x} = \mathbf{0}$ is satisfied for every element $\mathbf{x} \in R$.
4. *The equality* $\lambda \cdot \mathbf{0} = \mathbf{0}$ *is satisfied for any real number* λ *and* $\mathbf{0} \in R$.
5. *The equality* $\lambda\mathbf{x} = \mathbf{0}$ *yields one of the two equalities:* $\lambda = 0$ *or* $\mathbf{x} = \mathbf{0}$.
6. *The element* $(-1) \cdot \mathbf{x}$ *is the opposite of the element* $\mathbf{x}$.

461. There is a set of various systems of real numbers $(\xi_1; \xi_2; \ldots; \xi_n)$, $(\eta_1; \eta_2; \ldots; \eta_n)$, $(\zeta_1; \zeta_2; \ldots; \zeta_n)$, ... The sum of any two elements is specified by the equality

$$(\xi_1; \xi_2; \ldots; \xi_n) + (\eta_1; \eta_2; \ldots; \eta_n) = (\xi_1 + \eta_1; \xi_2 + \eta_2; \ldots; \xi_n + \eta_n),$$

and the product of any element by any number, by the equality

$$\lambda(\xi_1; \xi_2; \ldots; \xi_n) = (\lambda\xi_1; \lambda\xi_2; \ldots; \lambda\xi_n).$$

Prove that the set is a linear space.

Solution. Use the notation $\mathbf{x} = (\xi_1; \xi_2; \ldots; \xi_n)$, $\mathbf{y} = (\eta_1; \eta_2; \ldots; \eta_n)$, $\mathbf{z} = (\zeta_1; \zeta_2; \ldots; \zeta_n)$, ... Verify whether the conditions (1°)-(8°) are fulfilled.

(1°) $\mathbf{x} + \mathbf{y} = (\xi_1 + \eta_1; \xi_2 + \eta_2; \ldots; \xi_n + \eta_n)$, $\mathbf{y} + \mathbf{x} = (\eta_1 + \xi_1; \eta_2 + \xi_2; \ldots \ldots; \eta_n + \xi_n)$, i.e. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.

(2°) $\mathbf{x} + \mathbf{y} = (\xi_1 + \eta_1; \xi_2 + \eta_2; \ldots; \xi_n + \eta_n)$, $\mathbf{y} + \mathbf{z} = (\eta_1 + \zeta_1; \eta_2 + \zeta_2; \ldots; \eta_n + \zeta_n)$, $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = (\xi_1 + \eta_1 + \zeta_1; \xi_2 + \eta_2 + \zeta_2; \ldots; \xi_n + \eta_n + \zeta_n)$, $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\xi_1 + \eta_1 + \zeta_1; \xi_2 + \eta_2 + \zeta_2; \ldots; \xi_n + \eta_n + \zeta_n)$. Thus, $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.

(3°) The null-element is $\mathbf{0} = (0; 0; \ldots; 0)$. Indeed, $\mathbf{x} + \mathbf{0} = (\xi_1 + 0; \xi_2 + 0; \ldots; \xi_n + 0) = \mathbf{x}$.

(4°) The element $(-\xi_1; -\xi_2; \ldots; -\xi_n)$ is the inverse of the element $(\xi_1; \xi_2; \ldots; \xi_n)$, since $(\xi_1; \xi_2; \ldots; \xi_n) + (-\xi_1; -\xi_2; \ldots; -\xi_n) = (0; 0; \ldots; 0) = \mathbf{0}$.

(5°) $1 \cdot \mathbf{x} = (1 \cdot \xi_1; 1 \cdot \xi_2; \ldots; 1 \cdot \xi_n) = \mathbf{x}$.

(6°) $\lambda(\mu\mathbf{x}) = \lambda(\mu\xi_1; \mu\xi_2; \ldots; \mu\xi_n) = (\lambda\mu\xi_1; \lambda\mu\xi_2; \ldots; \lambda\mu\xi_n) = (\lambda\mu)\mathbf{x}$.

(7°) $(\lambda + \mu)\mathbf{x} = ((\lambda + \mu)\xi_1; (\lambda + \mu)\xi_2; \ldots; (\lambda + \mu)\xi_n) = (\lambda\xi_1 + \mu\xi_1; \lambda\xi_2 + \mu\xi_2; \ldots; \lambda\xi_n + \mu\xi_n) = (\lambda\xi_1; \lambda\xi_2; \ldots; \lambda\xi_n) + (\mu\xi_1; \mu\xi_2; \ldots; \mu\xi_n) = \lambda(\xi_1; \xi_2; \ldots; \xi_n) + \mu(\xi_1; \xi_2; \ldots; \xi_n) = \lambda\mathbf{x} + \mu\mathbf{x}$.

(8°) $\lambda(\mathbf{x} + \mathbf{y}) = \lambda(\xi_1 + \eta_1; \xi_2 + \eta_2; \ldots; \xi_n + \eta_n) = (\lambda\xi_1 + \lambda\eta_1; \lambda\xi_2 + \lambda\eta_2; \ldots; \lambda\xi_n + \lambda\eta_n) = (\lambda\xi_1; \lambda\xi_2; \ldots; \lambda\xi_n) + (\lambda\eta_1; \lambda\eta_2; \ldots; \lambda\eta_n) = \lambda(\xi_1; \xi_2; \ldots; \xi_n) + \lambda(\eta_1; \eta_2; \ldots; \eta_n) = \lambda\mathbf{x} + \lambda\mathbf{y}$.

462. Prove that the set of all complex numbers is a linear space.

463. Find whether the set of the systems of four real numbers $(\xi_1; \xi_2; 0; 0)$, $(\eta_1; \eta_2; 0; 0)$, $(\zeta_1; \zeta_2; 0; 0)$, where $\xi_1, \xi_2, \eta_1, \eta_2, \zeta_1, \zeta_2$ are various real numbers, is a linear space. Addition of elements and multiplication by a real number are defined as in problem 461.

464. Does the set of the elements $(\xi_1; \xi_2; 1; 1)$, $(\eta_1; \eta_2; 1; 1)$, $(\zeta_1; \zeta_2; 1; 1)$ form a linear space?

465. Is the set of various second-degree polynomials $\alpha_0 t^2 + \alpha_1 t + \alpha_2$, $\beta_0 t^2 + \beta_1 t + \beta_2$, $\gamma_0 t^2 + \gamma_1 t + \gamma_2$, ... a linear space?

466. Does the set of all polynomials of the degree not higher than the third form a linear space?

467. Given the functions $f_1(t), f_2(t), f_3(t)$, ... Is the set of these functions a linear space if the functions form: (1) a collection of all continuous functions on the closed interval $[a, b]$; (2) a collection of all differentiable functions on the interval $[a, b]$; (3) a collection of all elementary functions; (4) a collection of all nonelementary functions?

468. Given the set of various pairs of positive numbers: $\mathbf{x} = (\xi_1; \xi_2)$, $\mathbf{y} = (\eta_1; \eta_2)$, $\mathbf{z} = (\zeta_1; \zeta_2)$. Is the set a linear space if addition of two elements is specified by the equality $\mathbf{x} + \mathbf{y} = (\xi_1\eta_1; \xi_2\eta_2)$, and multiplication by a real number by the equality $\lambda\mathbf{x} = (\xi_1^\lambda; \xi_2^\lambda)$?

469. Can a linear space consist of (1) one vector; (2) two distinct vectors?

470. A vector **x** has been eliminated from the linear space. Can the set of vectors obtained as a result of the elimination remain a linear space?

471. An infinite number of vectors have been removed from the linear space. Can the set of vectors obtained as a result of the removal be a linear space?

472. The reserve of the dining-car is daily supplied with (1) sugar; (2) tea; (3) biscuits; (4) cakes; (5) fruit, to be sold to the passengers. Suppose $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5$ are the daily increments of these foodstuffs, in kg, respectively. If $\xi_i > 0$, then supply of the corresponding foodstuff exceeded the quantity sold that day and if $\xi_i < 0$, then the quantity sold exceeded the supply.

Is the collection of the systems of numbers $(\xi_1; \xi_2; \xi_3; \xi_4; \xi_5)$ a linear space? What is the meaning of the vector $(-100; 5; 0; -200; 3)$?

473. Does the collection of the triplets of integers $(\xi_1; \xi_2; \xi_3)$ form a linear space?

474. The rolling-stock of a depot is supplied daily with carriages of various kinds: luggage vans, mail-vans, hard-seated carriages, soft-seated carriages and sleeping-cars, which are used to form ordinary long-distance trains and express trains departing every day. Suppose $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5$ are the increments of the number of the respective carriages for a twenty-four-hour period. Is the collection of the numbers $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$ a linear space?

475. Is a space formed by all geometric vectors initiating at the origin and located in the first octant linear?

476. Prove that the set of all the solutions of the system of homogeneous linear equations

$$\begin{cases} a_1x + b_1y + c_1z = 0, \\ a_2x + b_2y + c_2z = 0 \end{cases}$$

forms a linear space.

Hint. Prove that if $(x_1; y_1; z_1)$ and $(x_2; y_2; z_2)$ are the solutions of the system, then $(x_1 + x_2; y_1 + y_2; z_1 + z_2)$ and $(\lambda x_1; \lambda y_1; \lambda z_1)$ are also the solutions of the system for any λ.

477. Prove that all the functions $y_1(x), y_2(x), y_3(x), \ldots$, which satisfy the differential equation $A_0y^{(n)} + A_1y^{(n-1)} + A_ny = 0$ ($A_0, A_1, \ldots, A_n$ being the functions of x) form a linear space.

5.1.2. Linearly independent vectors. Assume that $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots, \mathbf{u}$ are some vectors of the linear space R. The vector specified by the equality

$$\mathbf{v} = \alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z} + \ldots + \lambda\mathbf{u},$$

where $\alpha, \beta, \gamma, \ldots, \lambda$ are real numbers, also belongs to the linear space R. This vector is known as the *linear combination* of the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots, \mathbf{u}$.

Suppose the linear combination of the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots, \mathbf{u}$ is a null-vector, that is

$$\alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z} + \ldots + \lambda\mathbf{u} = 0. \tag{1}$$

If equality (1) can be also satisfied in the case when not all the numbers $\alpha, \beta, \gamma, \ldots, \lambda$ are equal to zero, the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots, \mathbf{u}$ are said to be *linearly dependent*.

The vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots, \mathbf{u}$ are called *linearly independent* if equality (1) is satisfied only for $\alpha = \beta = \gamma \ldots = \lambda = 0$. It is easy to prove that the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots, \mathbf{u}$ are linearly independent if and only if one of them can be represented as a linear combination of the other vectors.

478. Show that if there is a null-vector among the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots, \mathbf{u}$, then the vectors in question are linearly dependent.

Solution. Assume $\mathbf{x} = \mathbf{0}$. Since the equality

$$\alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z} + \ldots + \lambda\mathbf{u} = \mathbf{0}$$

can be satisfied only at $\alpha \neq 0, \beta = \gamma = \ldots = \lambda = 0$, the vectors are linearly dependent.

479. The elements of the linear space are the systems of the ordered real numbers $\mathbf{x}_i = (\xi_{1i}, \xi_{2i}, \ldots, \xi_{ni})$ $(i = 1, 2, 3, \ldots)$. What condition must be fulfilled by the numbers ξ_{ik} $(i = 1, 2, \ldots, n; k = 1, 2, \ldots, n)$ for the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ to be linearly independent if the sum of vectors and the product of a vector by a number are specified by the equalities $\mathbf{x}_i + \mathbf{x}_k = (\xi_{1i} + \xi_{1k}; \xi_{2i} + \xi_{2k}; \ldots; \xi_{ni} + \xi_{nk})$, $\lambda\mathbf{x}_i = (\lambda\xi_{1i}; \lambda\xi_{2i}; \ldots; \lambda\xi_{ni})$?

Solution. Let us consider the equation $\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \ldots + \alpha_n\mathbf{x}_n = \mathbf{0}$. It is equivalent to the system of equations

$$\begin{cases} \alpha_1\xi_{11} + \alpha_2\xi_{12} + \ldots + \alpha_n\xi_{1n} = 0, \\ \alpha_1\xi_{21} + \alpha_2\xi_{22} + \ldots + \alpha_n\xi_{2n} = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_1\xi_{n1} + \alpha_2\xi_{n2} + \ldots + \alpha_n\xi_{nn} = 0. \end{cases}$$

In the case the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly independent, the system must possess the unique solution $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$, that is

$$\begin{vmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \xi_{n1} & \xi_{n2} & \cdots & \xi_{nn} \end{vmatrix} \neq 0.$$

In particular, the vectors $(\xi_{11}; \xi_{21})$ and $(\xi_{12}; \xi_{22})$ are linearly independent if and only if $\xi_{11}\xi_{22} - \xi_{12}\xi_{21} \neq 0$.

480. Consider a linear space of polynomials of the degree not higher than the second. Prove that the vectors $\mathbf{P}_1 = 1 + 2t + 3t^2$, $\mathbf{P}_2 = 2 + 3t + 4t^2$ and $\mathbf{P}_3 = 3 + 5t + 7t^2$ are linearly dependent.

Solution. In this case it is immediately seen that $\mathbf{P}_3 = 1 \cdot \mathbf{P}_1 + 1 \cdot \mathbf{P}_2$; consequently, the vectors $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ are linearly dependent.

481. In what case are the vectors $\mathbf{x} = (\xi_1; \xi_2)$ and $\mathbf{y} = (\eta_1; \eta_2)$, defined in the condition of problem 468, linearly dependent?

Solution. It follows from the equality $\mathbf{x} = \lambda\mathbf{y}$ that $(\xi_1; \xi_2) = \lambda(\eta_1; \eta_2)$ or $(\xi_1; \xi_2) = (\eta_1^\lambda; \eta_2^\lambda)$, i.e. $\xi_1 = \eta_1^\lambda; \xi_2 + \eta_2^\lambda$. Hence we infer that $\ln \xi_1 \cdot \ln\eta_2 = \ln\eta_1 \cdot \cdot \ln \xi_2$.

482. Prove that three coplanar vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly dependent.

Hint. Reduce the vectors to a common origin and resolve one of them into components collinear to the two other vectors, respectively.

483. Prove that three noncoplanar vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent.

484. Prove that any four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are linearly dependent.

Solution. If three of the four vectors are coplanar, the problem is easy to solve. Assume that these three vectors are not coplanar. Let us reduce the four vectors to the common origin O. We construct a parallelepiped whose diagonal is the vector $\mathbf{d}$ and whose edges contain the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. It is easy to see that $\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$.

485. Prove that if n vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots, \mathbf{u}$ of a linear space are linearly dependent, then $n + 1$ vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots, \mathbf{u}, \mathbf{v}$ of that space are also linearly dependent.

5.1.3. Dimensionality and the basis of a linear space. If there are n linearly independent vectors in the linear space R but any $n + 1$ vectors of that space are linearly dependent, the space R is said to be *n-dimensional.* It is also customary to say that the dimensionality of the space R is equal to n and write $d(R) = n$. The space containing infinitely many linearly independent vectors is called *infinite-dimensional.* If R is an infinite-dimensional space, then $d(R) = \infty$.

A collection of n linearly independent vectors of an n-dimensional linear space is known as the *basis.* The following **theorem** holds true: *every vector of a linear n-dimensional space can be uniquely represented as a linear combination of the vectors of the basis.* Thus, if $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ is the basis of an n-dimensional linear space R, then any vector $\mathbf{x} \in R$ can be uniquely represented in the form

$$\mathbf{x} = \xi_1\mathbf{e}_1 + \xi_2\mathbf{e}_2 + \ldots + \xi_n\mathbf{e}_n.$$

Thus, the vector $\mathbf{x}$ in the basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ can be uniquely defined with the aid of the numbers $\xi_1, \xi_2, \ldots \xi_n$. These numbers are called the *coordinates* of the vector $\mathbf{x}$ in the given basis.

If $\mathbf{x} = \xi_1\mathbf{e}_1 + \xi_2\mathbf{e}_2 + \ldots + \xi_n\mathbf{e}_n$, $\mathbf{y} = \eta_1\mathbf{e}_1 + \eta_2\mathbf{e}_2 + \ldots + \eta_n\mathbf{e}_n$, then

$$\mathbf{x} + \mathbf{y} = (\xi_1 + \eta_1)\mathbf{e}_1 + (\xi_2 + \eta_2)\mathbf{e}_2 + \ldots + (\xi_n + \eta_n)\mathbf{e}_n,$$

$$\lambda\mathbf{x} = \lambda\xi_1\mathbf{e}_1 + \lambda\xi_2\mathbf{e}_2 + \ldots + \lambda\xi_n\mathbf{e}_n.$$

To determine the dimensionality of a linear space, it is useful to bear in mind the following **theorem:** *if any vector of the linear space R can be represented as a linear combination of the linearly independent vectors* $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, *then* $d(R) = n$ (and, consequently, *the vectors* $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ *form a basis in the space R*).

486. Given a linear space of various pairs of the ordered real numbers $\mathbf{x}_1 = (\xi_{11}; \xi_{21})$, $\mathbf{x}_2 = (\xi_{12}; \xi_{22})$, $\mathbf{x}_3 = (\xi_{13}; \xi_{23}), \ldots$, and addition of vectors and multiplication

of a vector by a real number are determined by the equalities $\mathbf{x}_i + \mathbf{x}_k = (\xi_{1i} + \xi_{1k}; \xi_{2i} + \xi_{2k})$; $\lambda\mathbf{x}_i = (\lambda\xi_{1i}; \lambda\xi_{2i})$. Prove that the vectors $\mathbf{e}_1 = (1; 2)$ and $\mathbf{e}_2 = (3; 4)$ form the basis of the given linear space. Find the coordinates of the vector $\mathbf{x} = (7; 10)$ in this basis.

Solution. The vectors $\mathbf{e}_1 = (1; 2)$ and $\mathbf{e}_2 = (3; 4)$ are linearly independent (see problem 479). Consider an arbitrary vector $\mathbf{y} = (\eta_1; \eta_2)$. Show that for any η_1 and η_2 there are numbers λ and μ such that the equality $\mathbf{y} = \lambda\mathbf{e}_1 + \mu\mathbf{e}_2$ or $(\eta_1; \eta_2) = (\lambda + 3\mu; 2\lambda + 4\mu)$ is satisfied.

It is easy to see that there exists a unique pair of values $(\lambda; \mu)$ for which this equality is satisfied. This follows from the fact that the system of equations

$$\begin{cases} \lambda + 3\mu = \eta_1, \\ 2\lambda + 4\mu = \eta_2 \end{cases}$$

is determinate.

Thus, the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ form a basis. Determine the coordinates of the vector $\mathbf{x} = (7; 10)$ in this basis. The problem reduces to determining λ and μ from the system of equations

$$\begin{cases} \lambda + 3\mu = 7, \\ 2\lambda + 4\mu = 10. \end{cases}$$

This yields $\lambda = 1$, $\mu = 2$, i.e. $\mathbf{x} = \mathbf{e}_1 + 2\mathbf{e}_2$.

487. Show that a linear space whose elements are the vectors $\mathbf{x} = (\xi_1; \xi_2; \ldots; \xi_n)$ (see problem 479) has the collection of vectors $\mathbf{e}_1 = (1; 0; 0; \ldots; 0)$, $\mathbf{e}_2 = (0; 1; 0; \ldots; 0)$, $\mathbf{e}_3 = (0; 0; 1; \ldots; 0)$, $\ldots$, $\mathbf{e}_n = (0; 0; 0; \ldots; 1)$ as its basis.

Solution. It is easy to see that

$$\mathbf{x} = \xi_1(1; 0; 0; \ldots; 0) + \xi_2(0; 1; 0; \ldots; 0) + \ldots + \xi_n(0; 0; 0; \ldots; 1),$$

i.e. $\mathbf{x} = \xi_1\mathbf{e}_1 + \xi_2\mathbf{e}_2 + \ldots + \xi_n\mathbf{e}_n$. Thus any vector can be represented as a linear combination of the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, and it follows that these vectors form a basis and the space R is n-dimensional.

488. What elements constitute the linear space with the basis $1, t, t^2, \ldots, t^{n-1}, t^n$ if addition of the elements and multiplication of an element by a real number are understood in the ordinary sense?

489. Show that the set of all second-order matrices is a four-dimensional linear space.

490. Show that the matrices $\mathbf{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, $\mathbf{e}_3 = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}$, $\mathbf{e}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$ form the basis of the linear space considered in problem 489.

491. Show that the elements $\mathbf{e}_1 = (1; 10)$ and $\mathbf{e}_2 = (10; 1)$ of the linear space considered in problem 468 form a basis. Find the coordinates of the vector $\mathbf{x} = (2; 3)$ in this basis.

Solution. Since $\ln 1 \cdot \ln 1 - \ln 10 \cdot \ln 10 \neq 0$, the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are linearly independent (see problem 481). Represent an arbitrary vector $\mathbf{y} = (\eta_1; \eta_2)$ as a linear combination of the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. Show that there exists a pair of numbers $(\lambda; \mu)$ such that the equality $\mathbf{y} = \lambda\mathbf{e}_1 + \mu\mathbf{e}_2$ is satisfied, or $(\eta_1; \eta_2) = (1^\lambda \cdot 10^\mu; 10^\lambda \cdot 1^\mu)$.

Consequently, $\mu = \log\eta_1$, $\lambda = \log\eta_2$. In particular, $\mathbf{x} = \mathbf{e}_1 \log 3 + \mathbf{e}_2 \log 2$. Thus, $(\log 3; \log 2)$ are the coordinates of the vector $\mathbf{x}$ in the basis $\mathbf{e}_1, \mathbf{e}_2$.

492. Show that the vectors $\mathbf{e}_1 = (1; 1; 1; \ldots; 1; 1)$, $\mathbf{e}_2 = (0; 1; 1; \ldots; 1; 1)$, $\mathbf{e}_3 = (0; 0; 1; \ldots; 1; 1)$, $\mathbf{e}_{n-1} = (0; 0; 0; \ldots; 1; 1)$, $\ldots$ $\mathbf{e}_n = (0; 0; 0; \ldots; 0; 1)$ can be taken as the basis of the n-dimensional space considered in problem 479.

Hint. Consider the vectors $\mathbf{e}'_1 = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{e}'_2 = \mathbf{e}_2 - \mathbf{e}_3$, $\ldots$, $\mathbf{e}'_{n-1} = \mathbf{e}_{n-1} - \mathbf{e}_n$, $\mathbf{e}'_n = \mathbf{e}_n$.

5.1.4. Isomorphism of linear spaces. Let us consider two linear spaces R and R'. The elements of the linear space R will be designated as $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ and the elements of the space R' *as* $\mathbf{x}', \mathbf{y}', \mathbf{z}', \ldots$.

The spaces R and R' are said to be *isomorphic* if a one-to-one correspondence $\mathbf{x} \leftrightarrow \mathbf{x}'$, $\mathbf{y} \leftrightarrow \mathbf{y}'$ can be established between their elements $\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}'$ such that $\mathbf{x} + \mathbf{y} \leftrightarrow \mathbf{x}' + \mathbf{y}'$, $\lambda\mathbf{x} \leftrightarrow \lambda\mathbf{x}'$($\lambda$ being any real number). An important theorem should be recalled which makes it easy to establish isomorphism of finite-dimensional linear spaces: *for two finite-dimensional spaces R and R' to be isomorphic it is necessary and sufficient that their dimensions be equal.*

493. Given two linear spaces R and R'. The elements of the space R are various differentiable functions of the argument t which vanish at $t = 0$. The elements of the space R' are the derivatives of the functions belonging to the space R. Prove that the spaces R and R' are isomorphic.

Solution. Assume that $f_1(t), f_2(t), f_3(t), \ldots$ are the functions of the space R and $\varphi_1(t), \varphi_2(t), \varphi_3(t), \ldots$ are the functions of the space R'. The fact that the functions possess indices does not imply that R and R' are countable sets.

Suppose $\varphi_i(t) = f'_i(t)$, then $f_i(t) = \int\limits_0^t \varphi_i(t)dt$.

Thus, a one-to-one correspondence is established between the elements of the linear spaces R and R' (the proof of their linearity is left to the reader).

The equalities

$$\varphi_i(t) + \varphi_k(t) = [f_i(t) + f_k(t)]', \quad f_i(t) + f_k(t) = \int\limits_0^t [\varphi_i(t) + \varphi_k(t)]dt,$$

$$\lambda\varphi_i(t) = [\lambda f_i(t)]', \quad \lambda f_i(t) = \int\limits_0^t \lambda\varphi_i(t)dt$$

are used to establish one-to-one correspondences

$$f_i(t) + f_k(t) \leftrightarrow \varphi_i(t) + \varphi_k(t), \quad \lambda f_i(t) \leftrightarrow \lambda\varphi_i(t).$$

Thus we see that R and R' are isomorphic spaces.

494. Prove that the sets of all geometric vectors and of polynomials of the degree not higher than the second are isomorphic linear spaces.

495. Given isomorphic linear spaces R and R'. One-to-one correspondences $\mathbf{x} \leftrightarrow$

$\leftrightarrow \mathbf{x}', \mathbf{y} \leftrightarrow \mathbf{y}', \ldots$ are established between the elements of these spaces. Prove that $\alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z} \leftrightarrow \alpha\mathbf{x}' + \beta\mathbf{y}' + \gamma\mathbf{z}'$ for any real α, β and γ.

496. Suppose R and R' are isomorphic linear spaces, with $\mathbf{x} \leftrightarrow \mathbf{x}'$. Prove that $(-\mathbf{x}) \leftrightarrow (-\mathbf{x}')$.

497. Given the isomorphic spaces R and R', $\mathbf{0}$ and $\mathbf{0}'$ being the null-elements of these spaces. Prove that $\mathbf{0} \leftrightarrow \mathbf{0}'$ irrespective of the way in which one-to-one correspondences are established between the other elements of these spaces.

498. Given various pairs of real numbers: $(\xi_1; \eta_1)$, $(\xi_2; \eta_2)$, $(\xi_3; \eta_3)$, ... Two linear spaces have been consructed: the space R with the elements $\mathbf{x}_1 = (\xi_1; \eta_1)$, $\mathbf{x}_2 = (\xi_2; \eta_2)$, $\mathbf{x}_3 = (\xi_3; \eta_3)$, . . ., in which addition of vectors and multiplication of a vector by a number are determined by the equalities $\mathbf{x}_1 + \mathbf{x}_2 = (\xi_1 + \xi_2; \eta_1 + \eta_2)$, $\lambda\mathbf{x}_1 = (\lambda\xi_1; \lambda\eta_1)$ and the space R' consisting of the vectors $\mathbf{x}_1' = (e^{-\xi_1}; e^{-\eta_1})$, $\mathbf{x}_2' = (e^{-\xi_2}; e^{-\eta_2})$, $\mathbf{x}_3' = (e^{-\xi_3}; e^{-\eta_3})$, . . ., in which the respective operations are determined by the equalities $\mathbf{x}_1' + \mathbf{x}_2' = (e^{-\xi_1\xi_2}; e^{-\eta_1-\eta_2})$, $\lambda\mathbf{x}_1' = (e^{-\lambda\xi_1}; e^{-\lambda\eta_1})$. Prove that the spaces R and R' are isomorphic.

499. Are the linear spaces R and R' isomorphic if the elements of R are the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ and the elements of R' are the vectors $2\mathbf{x}, 2\mathbf{y}, 2\mathbf{z}, \ldots$? Show that the spaces R and R' consist of the same elements.

5.2. Transformation of Coordinates upon a Transition to a New Basis

Suppose there exist two bases $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots$ (old one) and $\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3', \ldots$ (new one) in the n-dimensional linear space R^n. Given the relationships expressing each vector of the new basis in terms of the vectors of the old basis:

$$\begin{aligned} \mathbf{e}_1' &= a_{11}\mathbf{e}_1 + a_{12}\mathbf{e}_2 + \ldots + a_{n1}\mathbf{e}_n, \\ \mathbf{e}_2' &= a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2 + \ldots + a_{n2}\mathbf{e}_n, \\ \mathbf{e}_n' &= a_{1n}\mathbf{e}_1 + a_{2n}\mathbf{e}_2 + \ldots + a_{nn}\mathbf{e}_n. \end{aligned}$$

The matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ a_{n1} & a_{n2} \cdots a_{nn} \end{pmatrix}$$

is called the *transition matrix* from the old basis to the new one.

Let us take an arbitrary vector $\mathbf{x}$. Assume that $(\xi_1; \xi_2; \ldots; \xi_n)$ are the coordinates of this vector in the old basis, and $(\xi_1'; \xi_2'; \ldots; \xi_n')$ are its coordinates in the new basis. In this case the old coordinates of the vector $\mathbf{x}$ are expressed in terms of the new coordinates of this vector by the formulas

$$\begin{aligned} \xi_1 &= a_{11}\xi_1' + a_{12}\xi_2' + \ldots + a_{1n}\xi_n', \\ \xi_2 &= a_{21}\xi_1' + a_{22}\xi_2' + \ldots + a_{2n}\xi_n', \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \xi_n &= a_{n1}\xi_1' + a_{n2}\xi_2' + \ldots + a_{nn}\xi_n', \end{aligned}$$

which are called *formulas for transformation of coordinates.*

It is easy to see that the columns of the matrix A are the coordinates in the formulas of transition from the old basis to the new one, while the rows of the matrix are the coordinates in the formulas for transformation of the old coordinates in terms of the new ones.

500. Given the vectors $\mathbf{x} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4$. Resolve this vector with respect to the new basis $\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3', \mathbf{e}_4'$, if $\mathbf{e}_1' = \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4$, $\mathbf{e}_2' = \mathbf{e}_1 + \mathbf{e}_3 + \mathbf{e}_4$, $\mathbf{e}_3' = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4$, $\mathbf{e}_4' = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$.

Solution. Let us write out the matrix of transition from the old basis to the new one:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

The rows of this matrix are the coefficients in the formulas for transformation of coordinates:

$$\begin{aligned} \xi_1 &= \xi_2' + \xi_3' + \xi_4', \\ \xi_2 &= \xi_1' + \xi_3' + \xi_4', \\ \xi_3 &= \xi_1' + \xi_2' + \xi_4', \\ \xi_4 &= \xi_1' + \xi_2' + \xi_3'. \end{aligned}$$

Since $\xi_1 = \xi_2 = \xi_3 = \xi_4 = 1$, solution of the system of equations yields $\xi_1' = \xi_2' = \xi_3' = \xi_4' = 1/3$ and $\mathbf{x} = (1/3)(\mathbf{e}_1' + \mathbf{e}_2' + \mathbf{e}_3' + \mathbf{e}_4')$.

There is another method of solving the problem. Eliminating $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ from the system of equations

$$\begin{cases} \mathbf{x} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4, \\ \mathbf{e}_1' = 0 \cdot \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4, \\ \mathbf{e}_2' = \mathbf{e}_1 + 0 \cdot \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4, \\ \mathbf{e}_3' = \mathbf{e}_1 + \mathbf{e}_2 + 0 \cdot \mathbf{e}_3 + \mathbf{e}_4, \\ \mathbf{e}_4' = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + 0 \cdot \mathbf{e}_4, \end{cases}$$

we obtain

$$\begin{vmatrix} \mathbf{x} & 1 & 1 & 1 & 1 \\ \mathbf{e}_1' & 0 & 1 & 1 & 1 \\ \mathbf{e}_2' & 1 & 0 & 1 & 1 \\ \mathbf{e}_3' & 1 & 1 & 0 & 1 \\ \mathbf{e}_4' & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

It remains to expand this determinant with respect to the elements of the first column and express $\mathbf{x}$ in terms of $\mathbf{e}_1'$, $\mathbf{e}_2'$, $\mathbf{e}_3'$, $\mathbf{e}_4'$.

With due account of the peculiarities of the problem, we can suggest one more method of its solution. Since $\mathbf{e}_1' + \mathbf{e}_2' + \mathbf{e}_3' + \mathbf{e}_4' = 3\mathbf{e}_1 + 3\mathbf{e}_2 + 3\mathbf{e}_3 + 3\mathbf{e}_4$, it follows that $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 = (1/3)(\mathbf{e}_1' + \mathbf{e}_2' + \mathbf{e}_3' + \mathbf{e}_4')$. Hence $\mathbf{x} = (1/3)(\mathbf{e}_1' + \mathbf{e}_2' + \mathbf{e}_3' + \mathbf{e}_4')$.

501. Given the vector $\mathbf{x} = 8\mathbf{e}_1 + 6\mathbf{e}_2 + 4\mathbf{e}_3 - 18\mathbf{e}_4$. Resolve this vector with respect to the new basis related to the old basis by the equations $\mathbf{e}_1' = -3\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4$, $\mathbf{e}_2' = 2\mathbf{e}_1 - 4\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4$, $\mathbf{e}_3' = \mathbf{e}_1 + 3\mathbf{e}_2 - 5\mathbf{e}_3 + \mathbf{e}_4$, $\mathbf{e}_4' = \mathbf{e}_1 + \mathbf{e}_2 + 4\mathbf{e}_3 - 6\mathbf{e}_4$.

502. Given the vector $\mathbf{x} = 2(\mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_n)$. Resolve the vector $\mathbf{x}$ with respect to the basis $\mathbf{e}_1', \mathbf{e}_2', \ldots, \mathbf{e}_n'$ if $\mathbf{e}_1' = \mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{e}_2' = \mathbf{e}_2 + \mathbf{e}_3$, $\mathbf{e}_3' = \mathbf{e}_3 + \mathbf{e}_4$, $\ldots$, $\mathbf{e}_{n-1}' = \mathbf{e}_{n-1} + \mathbf{e}_n$, $\mathbf{e}_n' = \mathbf{e}_n + \mathbf{e}_1$.

503. The system of coordinates xOy has been turned about the origin through an angle α (Fig. 21). Express the coordinates of the vector $\mathbf{a} = x\mathbf{i} + y\mathbf{j}$ in the new system in terms of its coordinates in the old system.

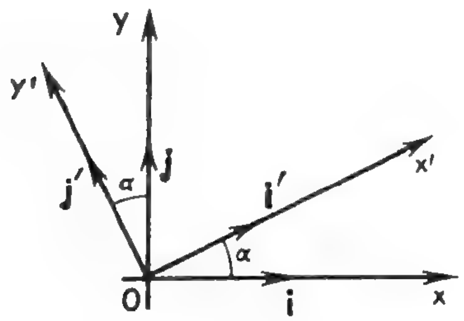

Fig. 21

Solution. Resolve the vectors $\mathbf{i}'$ and $\mathbf{j}'$ with respect to the unit vectors $\mathbf{i}$ and $\mathbf{j}$:

$$\mathbf{i}' = \mathbf{i}\cos\alpha + \mathbf{j}\sin\alpha,$$

$$\mathbf{j}' = \mathbf{i}\cos\left(\frac{\pi}{2} + \alpha\right) + \mathbf{j}\sin\left(\frac{\pi}{2} + \alpha\right).$$

Write the matrix of transition from the old basis $\mathbf{i}$, $\mathbf{j}$ to the new basis $\mathbf{i}'$, $\mathbf{j}'$:

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

Hence

$$x = x'\cos\alpha - y'\sin\alpha, \quad y = x'\sin\alpha + y'\cos\alpha,$$

that is

$$x' = x\cos\alpha + y\sin\alpha, \quad y' = -x\sin\alpha + y\cos\alpha.$$

504. Given the relationships $\mathbf{e}_1' = \alpha\mathbf{e}_2$, $\mathbf{e}_2' = \beta\mathbf{e}_3$, $\mathbf{e}_3' = \gamma\mathbf{e}_4$, $\mathbf{e}_4' = \delta\mathbf{e}_5$, $\mathbf{e}_5' = \varepsilon\mathbf{e}_1$.

Write the formulas relating the old coordinates $(\xi_1; \xi_2; \xi_3; \xi_4; \xi_5)$ of the vector $\mathbf{x}$ to the new coordinates $(\xi'_1; \xi'_2; \xi'_3; \xi'_4; \xi'_5)$ of the same vector.

505. Are the relationships $\mathbf{e}'_1 = \mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{e}'_2 = \mathbf{e}_3 - \mathbf{e}_1$, $\mathbf{e}'_3 = \mathbf{e}_1 - \mathbf{e}_2$ between the old basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and the new basis $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ possible?

5.3. Subspaces

5.3.1. A subspace of a linear space. A linear space R' is called a *subspace* of a linear space R if elements of the space R alone are the elements of the space R'.

For example, the set of all vectors parallel to one and the same plane is a subspace of all the geometric vectors of the space.

If $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots, \mathbf{u}$ are some vectors of the linear space R, then all the vectors $\alpha\mathbf{x} + \beta\mathbf{y} + \ldots + \lambda\mathbf{u}$, where $\alpha, \beta, \ldots, \lambda$ are various real numbers, form a subspace of the space R. The set of all linear combinations of the vectors $\alpha\mathbf{x} + \beta\mathbf{y} + \ldots + \lambda\mathbf{u}$ is called a *linear hull* of the vectors $\mathbf{x}, \mathbf{y}, \ldots, \mathbf{u}$ and is designated as $L(\mathbf{x}, \mathbf{y}, \ldots, \mathbf{u})$.

If R_1 is a subspace of the linear space R, then $d(R_1) \leqslant d(R)$.

Suppose there are two subspaces R_1 and R_2 of the linear space R.

The *intersection* of the subspaces R_1 and R_2 is the set R_3 of all the elements belonging simultaneously to R_1 and R_2. The notation $R_3 = R_1 \cap R_2$ means that R_3 is the intersection of subspaces R_1 and R_2. The *sum* of the subspaces R_1 and R_2 is the set R_4 of all elements of the form $\mathbf{x} + \mathbf{y}$, where $\mathbf{x} \in R_1$ and $\mathbf{y} \in R_2$. The notation $R_4 = R_1 + R_2$ signifies that the set R_4 is the sum of the subspaces R_1 and R_2.

It can be proved that the intersection R_3 and the sum R_4 are the subspaces of the space R. It should be borne in mind that

$$d(R_1) + d(R_2) = d(R_3) + d(R_4).$$

506. Can a subspace of the linear space R consist of one element?

507. Given the linear space R whose elements are various systems of real numbers: $\mathbf{x} = (\xi_1; \xi_2; \xi_3; \xi_4)$, $\mathbf{y} = (\eta_1; \eta_2; \eta_3; \eta_4)$, $\mathbf{z} = (\zeta_1; \zeta_2; \zeta_3; \zeta_4)$, ... Addition of two elements and multiplication of an element by a number are specified by the equations

$$\mathbf{x} + \mathbf{y} = (\xi_1 + \eta_1; \xi_2 + \eta_2; \quad \xi_3 + \eta_3; \xi_4 + \eta_4), \quad \lambda\mathbf{x} = (\lambda\xi_1; \lambda\xi_2; \lambda\xi_3; \lambda\xi_4).$$

Prove that the set R_1 of the elements $\mathbf{x}_1 = (0; \xi_2; \xi_3; \xi_4)$, $\mathbf{y}_1 = (0; \eta_2; \eta_3; \eta_4)$, $\mathbf{z}_1 = (0; \zeta_2; \zeta_3; \zeta_4)$, ... and the set R_2 of the elements $\mathbf{x}_2 = (\xi_1; 0; \xi_3; \xi_4)$, $\mathbf{y}_2 = (\eta_1; 0; \eta_3, \eta_4)$, $\mathbf{z}_2 = (\zeta_1; 0; \zeta_3; \zeta_4)$, ... are the subspaces of the linear space R.

508. Find the intersection R_3 and the sum R_4 of the subspaces R_1 and R_2 for the linear space R considered in problem 507.

509. Show that the equality $d(R_1) + d(R_2) = d(R_3) + d(R_4)$ is satisfied for the subspaces considered in problems 504 and 505.

510. Given a linear space consisting of all geometric vectors. Is the set of the vectors initiating at the origin and located in the first octant a subspace of that space?

511. Given the linear space R whose elements are the coordinates of the points $P = (x; y; z)$ of the first octant not lying on the coordinate planes. Addition of any two elements $P_1 = (x_1; y_1; z_1)$ and $P_2 = (x_2; y_2; z_2)$ is specified by the equation $P_1 + P_2 = (x_1x_2; y_1y_2; z_1z_2)$, and multiplication of the element $P = (x; y; z)$ by the real number λ, by the equation $\lambda P = (x^\lambda; y^\lambda; z^\lambda)$. Prove that the set R_1 of the points belonging to that space and located on the plane $z = 1$ is a subspace of the space R.

512. Given the linear space R of polynomials of the degree not higher than the fifth. Prove that the set R_1 of polynomials of the form $a_0t + a_1$ and the set R_2 of polynomials of the form $b_0t^4 + b_1t^2 + b_2$ are subspaces of the space R if addition of elements and multiplication of an element by a number are understood in the ordinary sense.

513. Using the condition of the previous problem, find the subspaces $R_3 = R_1 \cap R_2$ and $R_4 = R_1 + R_2$.

514. Consider two subspaces of the space R of all geometric vectors: R_1, the set of the vectors parallel to the coordinate plane xOy, and R_2, the set of the vectors parallel to the plane xOz. Find $R_3 = R_1 \cap R_2$ and $R_4 = R_1 + R_2$.

515. Assume that R_1 and R_2 are subspaces of the linear space R, and R_1' and R_2' are subspaces of the linear space R'. The subspaces R_1 and R_1' as well as R_2 and R_2' are known to be isomorphic. Prove that the subspaces $R_3 = R_1 \cap R_2$ and $R_3' = R_1' \cap R_2'$ as well as $R_4 = R_1 + R_2$ and $R_4' = R_1' + R_2'$ are isomorphic too.

516. Given the set of functions $f(x)$ continuous and positive on the closed interval $[-a, a]$. Prove that the set is a linear space if the product of the corresponding functions is assumed to be the sum of the vectors and the result of raising the corresponding functions to the power λ is taken to be the product of the vectors. Is the set of all the even function of that space a subspace? the set of all the odd functions of that space?

517. Consider a linear space of geometric vectors. Does the set of all the vectors $\mathbf{a} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$, where X, Y and Z are rational numbers, form a subspace of that space?

5.3.2. Subspaces formed as a result of solution of a homogeneous linear system of equations. Let us consider a homogeneous linear system of equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0. \end{cases} \tag{1}$$

Suppose that $x_1 = \lambda_1, x_2 = \lambda_2, \ldots, x_n = \lambda_n$ is some solution of the system. Write this solution in the form of a vector $\mathbf{f} = (\lambda_1; \lambda_2; \ldots; \lambda_n)$. A collection of linearly independent solutions $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n$ of the system of equations (1) is called a *fundamental system of solutions* if any solution of the system (1) can be represented as a linear combination of the vectors $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_k$.

Theorem on the existence of a fundamental system of solutions. *If the rank of the*

matrix

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

is smaller than n, system (1) *possesses nonzero solutions. The number of vectors defining the fundamental system of solutions can be found by the formula* $k = n - r$, *where* r *is the rank of the matrix.*

Thus it follows that if under consideration is the space R^n, whose vectors are various systems of n real numbers, then the collection of all the solutions of system (1) is a subspace of the space R^n. The dimensionality of this subspace is k.

518. Find the basis and the dimensionality of the subspace of solutions of the homogeneous linear system of equations

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 0, \\ (1/2)x_1 + x_2 + (3/2)x_3 + 2x_4 = 0, \\ (1/3)x_1 + (2/3)x_2 + x_3 + (4/3)x_4 = 0, \\ (1/4)x_1 + (1/2)x_2 + (3/4)x_3 + x_4 = 0. \end{cases}$$

Solution. The rank of the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1/2 & 1 & 3/2 & 2 \\ 1/3 & 2/3 & 1 & 4/3 \\ 1/4 & 1/2 & 3/4 & 1 \end{pmatrix}.$$

is equal to 1 since all the minors of the matrix, except for the first-degree minors, are equal to zero. The number of unknowns is four, therefore the dimensionality of the subspace of the solutions $k = n - r = 4 - 1 = 3$, that is, the subspace is three-dimensional. Since $r = 1$, it is sufficient to take some one equation of the system.

Let us take the first equation of the system and write it in the form $x_1 = -2x_2 - 3x_3 - 4x_4$. If $x_2 = 1, x_3 = 0, x_4 = 0$, then $x_1 = -2$; if $x_2 = 0, x_3 = 1, x_4 = 0$, then $x_1 = -3$; now if $x_2 = 0, x_3 = 0, x_4 = 1$, then $x_1 = -4$. Thus we have obtained the linearly independent vectors $\mathbf{f}_1 = (-2; 1; 0; 0)$, $\mathbf{f}_2 = (-3; 0; 1; 0)$, $\mathbf{f}_3 = (-4; 0; 0; 1)$, which form the basis of the three-dimensional subspace of solutions of the given system.

519. Show that the vector $\mathbf{f} = \mathbf{f}_1 - 2\mathbf{f}_2 + \mathbf{f}_3$ satisfies the system of equations (1).

520. Find the basis and the dimensionality of the subspace of solutions of the system of equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0, \\ 2x_1 - x_2 - x_3 = 0, \\ -2x_1 + 4x_2 - 2x_3 = 0. \end{cases}$$

Solution. The rank of the matrix

$$\begin{pmatrix} 1 & -2 & 1 \\ 2 & -1 & -1 \\ -2 & 4 & -2 \end{pmatrix}$$

is equal to 2 since the third-order determinant formed by the elements of the matrix is zero, and there are nonzero minors among the second-order minors. The dimensionality of the subspace of solutions $k = n - r = 3 - 2 = 1$. Since $r = 2$, it is sufficient to take two equations of the three given ones. Let us reject the third equation since its coefficients are proportional to the corresponding coefficients of the first equation.

Let us assume $x_3 = 1$ in the system

$$\begin{cases} x_1 - 2x_2 = -x_3, \\ 2x_1 - x_2 = x_3; \end{cases}$$

then the solution of the system

$$\begin{cases} x_1 - 2x_2 = -1, \\ 2x_1 - x_2 = 1 \end{cases}$$

is $x_1 = 1, x_2 = 1$.

Thus, we see that the subspace of solutions is defined by one base vector $\mathbf{f} = (1; 1; 1)$.

521. Find the dimensionality and the basis of the subspace of solutions of the following system of equations:

$$\begin{cases} x_1 + x_2 - x_3 + x_4 = 0, \\ x_1 - x_2 + x_3 - x_4 = 0, \\ 3x_1 + x_2 - x_3 + x_4 = 0, \\ 3x_1 - x_2 + x_3 - x_4 = 0. \end{cases}$$

Solution. Determine the rank of the matrix

$$A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 3 & 1 & -1 & 1 \\ 3 & -1 & 1 & -1 \end{pmatrix}.$$

Subtract the 2nd row from the 3rd, and the 1st from the 4th:

$$A \sim \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 2 & 2 & -2 & 2 \\ 2 & -2 & 2 & -2 \end{pmatrix}.$$

Since the elements of the 3rd row are proportional to the corresponding elements of the 1st row, and the elements of the 4th row to the elements of the 2nd row, the 3rd and 4th rows can be deleted:

$$A \sim \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Thus, the rank of the matrix A is equal to 2 and $k = n - r = 4 - 2 = 2$.

It follows that the dimensionality of the subspace of solutions is 2. Since $r = 2$, we shall take two equations of the given four:

$$\begin{cases} x_1 + x_2 - x_3 + x_4 = 0, \\ x_1 - x_2 + x_3 - x_4 = 0, \end{cases}$$

or

$$\begin{cases} x_1 + x_2 = x_3 - x_4, \\ x_1 - x_2 = -x_3 + x_4. \end{cases}$$

Putting $x_3 = 1, x_4 = 0$, we obtain a system

$$\begin{cases} x_1 + x_2 = 1, \\ x_1 - x_2 = -1. \end{cases}$$

Consequently, $x_1 = 0, x_2 = 1$, and $\mathbf{f}_1 = (0; 1; 1; 0)$.

Assuming now $x_3 = 0, x_4 = 1$, we have

$$\begin{cases} x_1 + x_2 = -1, \\ x_1 - x_2 = 1. \end{cases}$$

It follows that $x_1 = 0, x_2 = -1$, and $\mathbf{f}_2 = (0; -1; 0; 1)$.

The vectors $\mathbf{f}_1 = (0; 1; 1; 0)$, $\mathbf{f}_2 = (0; -1; 0; 1)$ may be assumed to be the base vectors of the subspace. The general solution of the system of equations is defined by the vector $\mathbf{f} = c_1\mathbf{f}_1 + c_2\mathbf{f}_2$, i.e. $\mathbf{f} = (0; c_1 - c_2; c_1; c_2)$.

522. Determine the dimensionality of the subspace of solutions, the basis and the general solution of the system of equations

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 0, \\ x_1 - 2x_2 + x_3 + x_4 - x_5 = 0. \end{cases}$$

5.4. Linear Transformations

5.4.1. Basic notions. A *transformation* A is said to be given in the linear space R if in accordance with some rule each vector $\mathbf{x} \in R$ is put into correspondence with the vector $\mathrm{A}x \in R$. The transformation A is called *linear* if the equalities

$$\mathrm{A}(\mathbf{x} + \mathbf{y}) = \mathrm{A}\mathbf{x} + \mathrm{A}\mathbf{y}, \quad \mathrm{A}(\lambda\mathbf{x}) = \lambda\mathrm{A}\mathbf{x}$$

are satisfied for any two vectors $\mathbf{x}$ and $\mathbf{y}$ and for any real number λ.

A linear transformation is called *identity transformation* if it maps any vector $\mathbf{x}$ into itself. An identity linear transformation is denoted by I. Thus, $\mathrm{I}\mathbf{x} = \mathbf{x}$.

523. Show that the transformation $\mathrm{A}\mathbf{x} = \alpha\mathbf{x}$, where α is a real number, is linear.

Solution. We have

$$\mathrm{A}(\mathbf{x} + \mathbf{y}) = \alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y} = \mathrm{A}\mathbf{x} + \mathrm{A}\mathbf{y},$$

$$\mathrm{A}(\lambda\mathbf{x}) = \alpha(\lambda\mathbf{x}) = \lambda(\alpha\mathbf{x}) = \lambda\mathrm{A}\mathbf{x}.$$

Thus, the two conditions defining a linear transformation are satisfied. The transformation A in question is known as a *similarity transformation*.

524. The transformation A in the linear space R is specified by the equality $\mathrm{A}\mathbf{x} = \mathbf{x} + \mathbf{x}_0$, where $\mathbf{x}_0 \in R$ is a fixed nonzero vector. Is the transformation A linear?

Solution. We infer from the equalities $\mathrm{A}\mathbf{x} = \mathbf{x} + \mathbf{x}_0$, $\mathrm{A}\mathbf{y} = \mathbf{y} + \mathbf{x}_0$, $\mathrm{A}(\mathbf{x} + \mathbf{y}) = \mathbf{x} + \mathbf{y} + \mathbf{x}_0$, $\mathrm{A}(\mathbf{x} + \mathbf{y}) = \mathrm{A}\mathbf{x} + \mathrm{A}\mathbf{y}$ that $\mathbf{x} + \mathbf{y} + \mathbf{x}_0 = (\mathbf{x} + \mathbf{x}_0) + (\mathbf{y} + \mathbf{x}_0)$. Hence it follows that $\mathbf{x}_0 = 0$, but this contradicts the hypothesis. Consequently, the transformation A is not linear.

525. Given a linear space of geometric vectors. The transformation A consists in replacing each vector by its component along the x-axis. Is the transformation linear?

Solution. Assume that $\mathbf{a} = X_1\mathbf{i} + Y_1\mathbf{j} + Z_1\mathbf{k}$ and $\mathbf{b} = X_2\mathbf{i} + Y_2\mathbf{j} + Z_2\mathbf{k}$ are arbitrary vectors and λ is an arbitrary real number. Since

$$\mathbf{a} + \mathbf{b} = (X_1 + X_2)\mathbf{i} + (Y_1 + Y_2)\mathbf{j} + (Z_1 + Z_2)\mathbf{k},\ \lambda\mathbf{a} = \lambda X_1\mathbf{i} + \lambda Y_1\mathbf{j} + \lambda Z_1\mathbf{k},$$

it follows that

$$\mathrm{A}(\mathbf{a} + \mathbf{b}) = (X_1 + X_2)\,\mathbf{i} = X_1\mathbf{i} + X_2\mathbf{i} = \mathrm{A}\mathbf{a} + \mathrm{A}\mathbf{b},\quad \mathrm{A}(\lambda\mathbf{a}) = \lambda X_1\mathbf{i} = \lambda\mathrm{A}\mathbf{a}.$$

Thus we see that A is a linear transformation.

526. Is the replacement of each geometric vector by its mirror image about the xOy coordinate plane a linear transformation?

527. Is multiplication of each geometric vector by its length a linear transformation?

528. In what case is the transformation A linear if $\mathrm{A}\mathbf{x} = \mathbf{x}_0$, where $\mathbf{x}_0$ is an arbitrary vector of the linear space R and $\mathbf{x}_0$ is a fixed vector?

529. Given the linear space of the vectors $\mathbf{x} = \xi_1\mathbf{e}_1 + \xi_2\mathbf{e}_2 + \xi_3\mathbf{e}_3 + \xi_4\mathbf{e}_4$, where $\xi_1, \xi_2, \xi_3, \xi_4$ are various real numbers. Suppose a is a fixed real number. Is the transformation A, specified by the equation $A\mathbf{x} = \xi_1\mathbf{e}_1 + \xi_2\mathbf{e}_2 + \xi_3\mathbf{e}_3 + \xi_4\mathbf{e}_4$, a linear transformation?

530. Given the linear space of the vectors $\mathbf{x} = \xi_1\mathbf{e}_1 + \xi_2\mathbf{e}_2 + \xi_3\mathbf{e}_3 + \xi_4\mathbf{e}_4$. The transformation A consists in an interchange of the second and third coordinates of each vector, that is, $\mathrm{A}\mathbf{x} = \xi_1\mathbf{e}_1 + \xi_3\mathbf{e}_2 + \xi_2\mathbf{e}_3 + \xi_4\mathbf{e}_4$. Is the transformation A linear?

531. Assume A to be a linear transformation. Prove that the transformation B, specified by the equation $\mathrm{B}\mathbf{x} = \mathrm{A}\mathbf{x} - 2\mathbf{x}$, is linear.

5.4.2. Matrix of a linear transformation. Suppose a linear transformation A is given in an n-dimensional linear space R whose basis is $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. Since $\mathrm{A}\mathbf{e}_1, \mathrm{A}\mathbf{e}_2, \ldots, \mathrm{A}\mathbf{e}_n$ are the vectors of the space R, each of them can be uniquely resolved with respect to the vectors of the basis:

$$\begin{aligned}
\mathrm{A}\mathbf{e}_1 &= a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2 + \ldots + a_{n1}\mathbf{e}_n,\\
\mathrm{A}\mathbf{e}_2 &= a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2 + \ldots + a_{n2}\mathbf{e}_n,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\mathrm{A}\mathbf{e}_n &= a_{1n}\mathbf{e}_1 + a_{2n}\mathbf{e}_2 + \ldots + a_{nn}\mathbf{e}_n.
\end{aligned}$$

The matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

is said to be the *matrix of a linear transformation* A in the basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. The columns of the matrix consist of the coefficients of the formulas for transformation of the base vectors. Let us take an arbitrary vector $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots + x_n\mathbf{e}_n$ in the space R. Since $\mathrm{A}\mathbf{x} \in R$, the vector $\mathrm{A}\mathbf{x}$ can be also resolved with respect to the vectors of the basis:

$$\mathrm{A}\mathbf{x} = x_1'\mathbf{e}_1 + x_2'\mathbf{e}_2 + \ldots + x_n'\mathbf{e}_n.$$

The coordinates $(x_1'; x_2'; \ldots; x_n')$ of the vector $\mathrm{A}\mathbf{x}$ can be expressed in terms of the coordinates $(x_1; x_2; \ldots; x_n)$ of the vector $\mathbf{x}$ by the formulas

$$\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n,\\
x_2' &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
x_n' &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n.
\end{aligned}$$

These n equalities can be called a linear transformation A in the basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. The coefficients in the formulas for this linear transformation are the elements of the rows of the matrix A.

532. Find the matrix of the identity transformation I in an n-dimensional space.

Solution. An identity transformation does not change the base vectors: $\mathbf{e}_1' = \mathbf{e}_1$, $\mathbf{e}_2' = \mathbf{e}_2$, $\mathbf{e}_3' = \mathbf{e}_3, \ldots, \mathbf{e}_n' = \mathbf{e}_n$, i.e.

$$\begin{aligned}
\mathbf{e}_1' &= 1\cdot\mathbf{e}_1 + 0\cdot\mathbf{e}_2 + \ldots + 0\cdot\mathbf{e}_n,\\
\mathbf{e}_2' &= 0\cdot\mathbf{e}_1 + 1\cdot\mathbf{e}_2 + \ldots + 0\cdot\mathbf{e}_n,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\mathbf{e}_n' &= 0\cdot\mathbf{e}_1 + 0\cdot\mathbf{e}_2 + \ldots + 1\cdot\mathbf{e}_n.
\end{aligned}$$

Consequently, the matrix of the linear transformation is the unit matrix

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & \dots 0 \\ 0 & 1 & 0 & \dots 0 \\ 0 & 0 & 1 & \dots 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots 1 \end{pmatrix}.$$

533. Find the matrix of the similarity transformation $\mathbf{Ax} = \alpha\mathbf{x}$ in an n-dimensional space.

534. A linear transformation A is considered in a 4-dimensional linear space. Write this transformation in a coordinate form if $\mathbf{Ae}_1 = \mathbf{e}_3 + \mathbf{e}_4$, $\mathbf{Ae}_2 = \mathbf{e}_1 + \mathbf{e}_4$, $\mathbf{Ae}_3 = \mathbf{e}_4 + \mathbf{e}_2$, $\mathbf{Ae}_4 = \mathbf{e}_2 + \mathbf{e}_3$.

Solution. The transformation matrix A has the form

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Consequently, the transformation A in a coordinate form is written as follows: $x_1' = x_2 + x_3, x_2' = x_3 + x_4, x_3' = x_1 + x_4, x_4' = x_1 + x_2$.

535. A linear transformation of the collection of all vectors on the xOy plane consists in the rotation of each vector counterclockwise through an angle α (Fig. 22). Find the matrix of this linear transformation in a coordinate form.

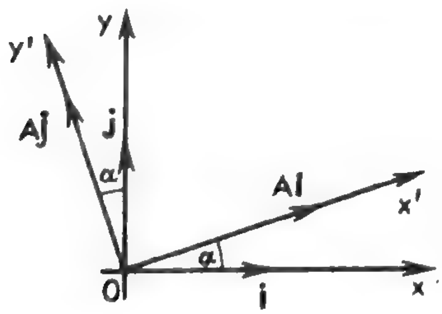

Fig. 22

Solution. Since

$$\mathbf{Ai} = \mathbf{i}\cos\alpha + \mathbf{j}\sin\alpha, \quad \mathbf{Aj} = -\mathbf{i}\sin\alpha + \mathbf{j}\cos\alpha,$$

it follows that

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

Thus, the linear transformation in question has the form

$$x' = x\cos\alpha - y\sin\alpha; \quad y' = x\sin\alpha + y\cos\alpha.$$

536. Consider the linear space of the vectors $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 + x_4\mathbf{e}_4$, where x_1, x_2, x_3, x_4 are various real numbers. Prove that the transformation A, specified by the equality $\mathrm{A}\mathbf{x} = x_2\mathbf{e}_1 + x_3\mathbf{e}_2 + x_4\mathbf{e}_3 + x_1\mathbf{e}_4$ is linear, and find its matrix.

5.4.3. Operations on linear transformations. We shall use the following notation in the definitions that will follow: A and B are arbitrary linear transformations in a linear space R, λ is an arbitrary real number, $\mathbf{x} \in R$ is any element.

The *sum of the linear transformations* A and B is a transformation C_1 specified by the equality $\mathrm{C}_1\mathbf{x} = \mathrm{A}\mathbf{x} + \mathrm{B}\mathbf{x}$. The notation is $\mathrm{C}_1 = \mathrm{A} + \mathrm{B}$.

The *product of the linear transformation* A *by the number* λ is a transformation C_2 specified by the equality $\mathrm{C}_2\mathbf{x} = \lambda\mathrm{A}\mathbf{x}$. The notation is $\mathrm{C}_2 = \lambda\mathrm{A}$.

The *product of the linear transformation* A *by the linear transformation* B is a transformation C_3 specified by the equality $\mathrm{C}_3\mathbf{x} = \mathrm{AB}\mathbf{x}$. The notation is $\mathrm{C}_3 = \mathrm{AB}$.

The transformations C_1, C_2, C_3 are linear. The matrices of the linear transformations C_1, C_2, C_3 are determined by the equations $C_1 = A + B$, $C_2 = \lambda A$, $C_3 = AB$.

Summation of linear transformations is commutative, while the product AB differs, in general, from the product BA.

Here are some properties of the operations that can be performed on linear transformations in the space R:

$$\mathrm{A(BC) = (AB)C; \quad AI = IA = A; \quad (A + B)C = AC + BC;}$$
$$\mathrm{C(A + B) = CA + CB.}$$

If for the linear transformation A there can be found transformations B and C such that BA = E, AC = I, then B = C. In this case the notation is $\mathrm{B} = \mathrm{C} = \mathrm{A}^{-1}$, and the linear transformation A^{-1} is called an *inverse linear transformation* with respect to the linear transformation A. Thus, $\mathrm{A}^{-1}\mathrm{A} = \mathrm{A}\mathrm{A}^{-1} = \mathrm{I}$.

The linear transformation A in a finite-dimensional space is said to be *nonsingular* if the determinant of the transformation matrix is nonzero. It should be borne in mind that every nonsingular linear transformation A has an inverse A^{-1} and the inverse is unique.

If the nonsingular linear transformation A in a coordinate form is specified by the equations

$$\begin{aligned} x' &= a_{11}x + a_{12}y + \ldots + a_{1n}u, \\ y' &= a_{21}x + a_{22}y + \ldots + a_{2n}u, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ u' &= a_{n1}x + a_{n2}y + \ldots + a_{nn}u, \end{aligned}$$

then the inverse linear transformation A^{-1} has the form

$$x = \frac{A_{11}}{|A|}x' + \frac{A_{21}}{|A|}y' + \ldots + \frac{A_{n1}}{|A|}u',$$

$$y = \frac{A_{12}}{|A|}x' + \frac{A_{22}}{|A|}y' + \ldots + \frac{A_{n2}}{|A|}u',$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$u = \frac{A_{1n}}{|A|}x' + \frac{A_{2n}}{|A|}y' + \ldots + \frac{A_{nn}}{|A|}u'.$$

Here A_{ij} is a cofactor of the element a_{ij} of the matrix A, $|A|$ being the determinant of the matrix A.

The matrix of the linear transformation A^{-1} is inverse to the matrix A and is specified by the equation

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} & \ldots & A_{n1} \\ A_{12} & A_{22} & \ldots & A_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ A_{1n} & A_{n2} & \ldots & A_{nn} \end{pmatrix}.$$

537. The transformation A consists in a rotation of every vector of the plane xOy through an angle $\alpha = \pi/4$. Find the transformation A + I in a coordinate form.

Solution. We have

$$\mathrm{A}\mathbf{i} = \mathbf{i}\cos(\pi/4) + \mathbf{j}\sin(\pi/4) = (\sqrt{2}/2)\mathbf{i} + (\sqrt{2}/2)\mathbf{j};$$

$$A\mathbf{j} = \mathbf{i}\cos(3\pi/4) + \mathbf{j}\sin(3\pi/4) = -(\sqrt{2}/2)\mathbf{i} + (\sqrt{2}/2)\mathbf{j}.$$

Consequently,

$$A = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}.$$

Since $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, it follows that

$$A + I = \begin{pmatrix} \sqrt{2}/2 + 1 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 + 1 \end{pmatrix}.$$

Thus, the linear transformation A + I can be written with the aid of the equalities

$$x' = (\sqrt{2}/2 + 1)x - (\sqrt{2}/2)y, \quad y' = (\sqrt{2}/2)x + (\sqrt{2}/2 + 1)y.$$

538. Given two linear transformations:

$$\begin{aligned} x' &= x + 2y + 3z, \\ y' &= 4x + 5y + 6z, \\ z' &= 7x + 8y + 9z \end{aligned} \quad \text{(A)} \quad \text{and} \quad \begin{aligned} x' &= x + 3y + 4.5z, \\ y' &= 6x + 7y + 9z, \\ z' &= 10.5x + 12y + 13z. \end{aligned} \quad \text{(B)}$$

Find 3A — 2B.

539. Given linear transformations:

$$\begin{aligned} x' &= x + y, \\ y' &= y + z, \\ z' &= z + x \end{aligned} \quad \text{(A)} \quad \text{and} \quad \begin{aligned} x' &= y + z, \\ y' &= x + z, \\ z' &= x + y. \end{aligned} \quad \text{(B)}$$

Find the transformations AB and BA.

Solution. The matrices of the given transformations have the form

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

We find the product of the matrices:

$$AB = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \quad BA = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

In the given case $AB = BA$ and, therefore, the linear transformations AB and BA coincide. The coordinate form of the transformation AB is written as follows:

$$x' = x + y + 2z, \quad y' = 2x + y + z, \quad z' = x + 2y + z.$$

540. Suppose the collection of vectors $\mathbf{u} = x\mathbf{i} + y\mathbf{j}$ on the xOy plane undergo two linear transformations: A, replacement of the vector by its component along the x-axis; B, a mirror reflection of the vector about the bisector of the first and third quadrants. Find the transformations AB and BA.

Solution. By the hypothesis, $A\mathbf{u} = x\mathbf{i}$, $B\mathbf{u} = x\mathbf{j} + y\mathbf{i}$. Thus, $A\mathbf{i} = \mathbf{i}$, $A\mathbf{j} = 0$, $B\mathbf{i} = \mathbf{j}$, $B\mathbf{j} = \mathbf{i}$, i.e.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad BA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Thus, the transformation AB is specified by the equations $x' = y$, $y' = 0$, and the transformation BA by the equations $x' = 0$, $y' = x$. We recommend the reader to use the geometric argument to obtain these equalities.

541. The transformation A consists in a rotation of every vector of the xOy plane through an angle α. Find the matrix of the transformation A^2 (i.e. $A \cdot A$).

Solution. Since $A\mathbf{i} = \mathbf{i}\cos\alpha + \mathbf{j}\sin\alpha$, $A\mathbf{j} = -\mathbf{i}\sin\alpha + \mathbf{j}\cos\alpha$, it follows that

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

Consequently,

$$A^2 = \begin{pmatrix} \cos^2\alpha - \sin^2\alpha & -2\sin\alpha\cos\alpha \\ 2\sin\alpha\cos\alpha & \cos^2\alpha - \sin^2\alpha \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{pmatrix}.$$

Thus, the transformation A^2 in coordinate form is specified by the equations

$$x' = x \cos 2\alpha - y \sin 2\alpha,$$
$$y' = x \sin 2\alpha + y \cos 2\alpha.$$

A purely geometric approach can also lead to these results.

542. The linear transformation A consists in a rotation of every vector of the xOy plane through an angle $\pi/4$. Find the matrix of the linear transformation $B = A^2 + \sqrt{2}A + I$.

Solution. We have

$$A = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} = \sqrt{2}/2 \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};$$

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

543. Given a space of geometric vectors. Suppose the linear transformation A is a rotation of the space about the Oz axis through an angle $\pi/4$ and the linear transformation B, a rotation of the space about the Ox axis through the same angle. Find the matrix of the linear transformation AB.

Solution. We have

$$\mathbf{Ai} = \mathbf{i} \cos\frac{\pi}{4} + \mathbf{j} \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j},$$

$$\mathbf{Aj} = -\mathbf{i} \sin\frac{\pi}{4} + \mathbf{j} \cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j},$$

$$\mathbf{Ak} = \mathbf{k};\ \mathbf{Bi} = \mathbf{i},\ \mathbf{Bj} = \frac{\sqrt{2}}{2}\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k},\ \mathbf{Bk} = -\frac{\sqrt{2}}{2}\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}.$$

Consequently,

$$A = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix},$$

$$AB = \begin{pmatrix} \sqrt{2}/2 & -1/2 & 1/2 \\ \sqrt{2}/2 & 1/2 & -1/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}.$$

544. Given the linear transformation A:

$$x' = -0.5(y+z), \quad y' = -0.5(x+z), \quad z' = -0.5(x+y).$$

Find the matrix of the inverse linear transformation.

545. Consider a collection of all geometric vectors. The linear transformation A is a mirror image of these vectors about the plane P. Find A^{-1}.

546. A linear transformation A is given in a linear space with the basis $\mathbf{e}_1, \mathbf{e}_2$. Find the matrix of the inverse transformation if $A\mathbf{e}_1 = \mathbf{e}_2$, $A\mathbf{e}_2 = \mathbf{e}_1$.

547. The linear transformation A consists in a rotation of every vector of the xOy plane through an angle α. Find the matrix $B = A + A^{-1}$.

548. Given the linear transformation A: $x' = x + y$, $y' = 2(x+y)$. Find the inverse linear transformation.

549. The linear transformation A consists in a rotation of every vector of the xOy plane through an angle $\pi/4$. Find the matrix A^{-2}.

550. At what value of λ does the linear transformation $x' = -2x + y + z$, $y' = x - 2y + z$, $z' = x + z + \lambda z$ have no inverse?

5.4.4. Characteristic numbers and eigenvectors of a linear transformation. Suppose R is a given n-dimensional linear space. The nonzero vector $\mathbf{x} \in R$ is called an *eigenvector* of the linear transformation A if there is a number λ such that the equality $A\mathbf{x} = \lambda\mathbf{x}$ is satisfied. The number λ is called a *characteristic number* of the linear transformation A corresponding to the vector $\mathbf{x}$.

If the linear transformation A in the basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ has the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

then the characteristic numbers of the linear transformation A are the real roots λ_1, $\lambda_2, \ldots \lambda_n$ of the nth degree equation which can be written in the form

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0.$$

It is known as a *characteristic equation* and its left-hand side, as a *characteristic polynomial* of the linear transformation A. The eigenvector $\mathbf{x}_k$ corresponding to the characteristic number λ_k is any vector $\xi_1\mathbf{e}_1 + \xi_2\mathbf{e}_2 + \ldots + \xi_n\mathbf{e}_n$ whose coordinates satisfy the system of homogeneous equations

$$\begin{cases} (a_{11} - \lambda_k)\xi_1 + a_{12}\xi_2 + \ldots + a_{1n}\xi_n = 0, \\ a_{21}\xi_1 + (a_{22} - \lambda_k)\xi_2' + \ldots + a_{2n}\xi_n = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}\xi_1 + a_{n2}\xi_2 + \ldots + (a_{nn} - \lambda_k)\xi_n = 0. \end{cases}$$

Note the following important theorems.

A characteristic polynomial of a linear transformation does not depend on the choice of the basis.

If the matrix A of the linear transformation A is symmetrical, then all the roots of the characteristic equation $|A - \lambda E| = 0$ *are real numbers.*

551. Find the characteristic numbers and the eigenvectors of the linear transformation A specified by the equations $x' = 5x + 4y$, $y' = 8x + 9y$.

Solution. The transformation matrix is written as follows:

$$A = \begin{pmatrix} 5 & 4 \\ 8 & 9 \end{pmatrix}.$$

The characteristic equation has the form

$$\begin{vmatrix} 5 - \lambda & 4 \\ 8 & 9 - \lambda \end{vmatrix} = 0, \quad \text{or} \quad \lambda^2 - 14\lambda + 13 = 0;$$

the characteristic numbers $\lambda_1 = 1$, $\lambda_2 = 13$.

To determine the coordinates of the eigenvectors, we obtain two systems of linear equations:

$$\begin{cases} (5 - \lambda_1)\xi_1 + 4\xi_2 = 0, \\ 8\xi_1 + (9 - \lambda_1)\xi_2 = 0; \end{cases} \qquad \begin{cases} (5 - \lambda_2)\xi_1 + 4\xi_2 = 0, \\ 8\xi_1 + (9 - \lambda_2)\xi_2 = 0. \end{cases}$$

Since $\lambda_1 = 1$, the first system can be written as follows:

$$\begin{cases} 4\xi_1 + 4\xi_2 = 0, \\ 8\xi_1 + 8\xi_2 = 0. \end{cases}$$

Thus, the values of ξ_1 and ξ_2 must satisfy the equation $\xi_1 + \xi_2 = 0$, or $\xi_2 = -\xi_1$. Consequently, the solution of the system has the form $\xi_1 = c_1$, $\xi_2 = -c_1$, where c_1 is an arbitrary value. Thus, the characteristic number $\lambda = 1$ is associated with a family of eigenvectors $\mathbf{u} = c_1\mathbf{e}_1 - c_1\mathbf{e}_2$, i.e. $\mathbf{u} = c_1(\mathbf{e}_1 - \mathbf{e}_2)$.

The value $\lambda_2 = 13$ leads to a system of equations

$$\begin{cases} -8\xi_1 + 4\xi_2 = 0, \\ 8\xi_1 - 4\xi_2 = 0, \end{cases}$$

i.e. $\xi_2 = 2\xi_1$. Putting $\xi_1 = c_2$, we obtain $\xi_2 = 2c_2$. Consequently, the characteristic number $\lambda = 13$ is associated with a family of eigenvectors $\mathbf{v} = c_2(\mathbf{e}_1 + 2\mathbf{e}_2)$.

Thus, assigning to the quantities c_1 and c_2 in the equations $\mathbf{u} = c_1(\mathbf{e}_1 - \mathbf{e}_2)$, $\mathbf{v} = c_2(\mathbf{e}_1 + 2\mathbf{e}_2)$ various numerical values, we obtain the eigenvectors of the linear transformation **A**.

552. Given a linear transformation with the matrix $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. Find the characteristic numbers and the eigenvectors of the transformation.

553. Find the characteristic numbers and the eigenvectors of a linear transformation with the matrix $A = \begin{pmatrix} 6 & -4 \\ 4 & -2 \end{pmatrix}$.

554. Find the characteristic numbers and the eigenvectors of a linear transformation with the matrix $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

555. Determine the characteristic numbers and the eigenvectors of a linear transformation with the matrix

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution. Let us derive a characteristic equation:

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0,$$

i.e. $(1-\lambda)[(2-\lambda)^2 - 1] = 0$, $(1-\lambda)^2(3-\lambda) = 0$, $\lambda_{1,2} = 1$, $\lambda_3 = 3$.

If $\lambda = 1$, then to determine the coordinates of the eigenvector, we obtain a system of equations

$$\begin{cases} \xi_1 - \xi_2 + \xi_3 = 0, \\ -\xi_1 + \xi_2 - \xi_3 = 0, \\ \xi_3 = 0. \end{cases}$$

Thus, the characteristic number $\lambda = 1$ is associated with the family of the eigenvectors $\mathbf{u} = c_1(\mathbf{e}_1 + \mathbf{e}_2)$.

If $\lambda = 3$, then to determine the coordinates of the eigenvector, we obtain a system of equations

$$\begin{cases} -\xi_1 - \xi_2 + \xi_3 = 0, \\ -\xi_1 - \xi_2 + \xi_3 = 0, \\ \xi_3 = 0. \end{cases}$$

The family of the eigenvectors corresponding to that characteristic number is specified by the equality $\mathbf{v} = c_2(\mathbf{e}_1 - \mathbf{e}_2)$.

556. Determine the characteristic numbers and the eigenvectors of a linear transformation A with the matrix

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}.$$

557. Prove that if

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is a symmetric matrix, and the real numbers α, β and γ are different from zero, then all the roots of the characteristic equation of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12}\alpha/\beta & a_{13}\alpha/\gamma \\ a_{21}\beta/\alpha & a_{22} & a_{23}\beta/\gamma \\ a_{31}\gamma/\alpha & a_{32}\gamma/\beta & a_{33} \end{pmatrix}$$

are real numbers.

Solution. Consider a linear transformation A with a matrix A in the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$:

$$\begin{aligned} \mathrm{A}\mathbf{e}_1 &= a_{11}\mathbf{e}_1 + (a_{21}\beta/\alpha)\mathbf{e}_2 + (a_{31}\gamma/\alpha)\mathbf{e}_3, \\ \mathrm{A}\mathbf{e}_2 &= (a_{12}\alpha/\beta)\mathbf{e}_1 + a_{22}\mathbf{e}_2 + (a_{32}\gamma/\beta)\mathbf{e}_3, \\ \mathrm{A}\mathbf{e}_3 &= (a_{13}\alpha/\gamma)\mathbf{e}_1 + (a_{23}\beta/\gamma)\mathbf{e}_2 + a_{33}\mathbf{e}_3 \end{aligned}$$

or

$$\begin{aligned} \mathrm{A}(\alpha\mathbf{e}_1) &= a_{11}\alpha\mathbf{e}_1 + a_{21}\beta\mathbf{e}_2 + a_{31}\gamma\mathbf{e}_3, \\ \mathrm{A}(\beta\mathbf{e}_2) &= a_{12}\alpha\mathbf{e}_1 + a_{22}\beta\mathbf{e}_2 + a_{32}\gamma\mathbf{e}_3, \\ \mathrm{A}(\gamma\mathbf{e}_3) &= a_{13}\alpha\mathbf{e}_1 + a_{23}\beta\mathbf{e}_2 + a_{33}\gamma\mathbf{e}_3. \end{aligned}$$

Putting $\alpha\mathbf{e}_1 = \mathbf{e}'_1$, $\beta\mathbf{e}_2 = \mathbf{e}'_2$, $\gamma\mathbf{e}_3 = \mathbf{e}'_3$, we obtain

$$\begin{aligned} \mathrm{A}\mathbf{e}'_1 &= a_{11}\mathbf{e}'_1 + a_{21}\mathbf{e}'_2 + a_{31}\mathbf{e}'_3, \\ \mathrm{A}\mathbf{e}'_2 &= a_{12}\mathbf{e}'_1 + a_{22}\mathbf{e}'_2 + a_{32}\mathbf{e}'_3, \\ \mathrm{A}\mathbf{e}'_3 &= a_{13}\mathbf{e}'_1 + a_{23}\mathbf{e}'_2 + a_{33}\mathbf{e}'_3. \end{aligned}$$

Thus, the matrix of the linear transformation A in the basis $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$ is the symmetric matrix

$$A' = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Consequently, the characteristic equation of the linear transformation A in the basis $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$ possesses only real roots. Since upon a transition to the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ the characteristic numbers do not change, the characteristic equation of the matrix A possesses the same roots.

558. The linear transformation A consists in a rotation of a space through an angle $\pi/3$ about the z-axis. Find the characteristic numbers and the eigenvectors of this transformation.

Hint. Show that the matrix of this transformation has the form

$$A = \begin{pmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

559. Knowing the characteristic numbers of the linear transformation A, find the characteristic numbers of the inverse linear transformation A^{-1}.

Hint. Show that the equation $|A^{-1} - \lambda I| = 0$ yields the equation $|A - (1/\lambda)I| = 0$.

560. Find the characteristic numbers and the eigenvectors of a linear transformation A with the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

561. Find the characteristic numbers and the eigenvectors of a linear transformation with the matrix

$$A = \begin{pmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{pmatrix}.$$

5.5. Euclidean Space

The linear space R is said to be *Euclidean* if there is a rule which makes it possible to construct, for every two vectors $\mathbf{x}$ and $\mathbf{y}$ belonging to R, a real number called a *scalar product* of the vectors $\mathbf{x}$ and $\mathbf{y}$ and designated as $(\mathbf{x}, \mathbf{y})$, the rule complying with the following conditions:

1°. $(\mathbf{x}, \mathbf{y}) = (\mathbf{y}, \mathbf{x})$.
2°. $(\mathbf{x}, \mathbf{y} + \mathbf{z}) = (\mathbf{x}, \mathbf{y}) + (\mathbf{x}, \mathbf{z})$.
3°. $(\lambda\mathbf{x}, \mathbf{y}) = \lambda(\mathbf{x}, \mathbf{y})$ for any real number λ.
4°. $(\mathbf{x}, \mathbf{x}) > 0$, if $\mathbf{x} \neq \mathbf{0}$.

It follows from conditions 1°-4° that

(a) $(\mathbf{y} + \mathbf{z}, \mathbf{x}) = (\mathbf{y}, \mathbf{x}) + (\mathbf{z}, \mathbf{x})$;
(b) $(\mathbf{x}, \lambda\mathbf{y}) = \lambda(\mathbf{x}, \mathbf{y})$;
(c) $(\mathbf{0}, \mathbf{x}) = 0$ for any vector $\mathbf{x}$.

The scalar product of any vector $\mathbf{x} \in R$ into itself is called a *scalar square* of the vector $\mathbf{x}$.

The *length* of the vector $\mathbf{x}$ in a Euclidean space is a square root of the scalar square of that vector, i.e. $|\mathbf{x}| = \sqrt{(\mathbf{x}, \mathbf{x})}$.

If λ is an arbitrary real number, and $\mathbf{x}$ is an arbitrary vector of a Euclidean space, then $|\lambda\mathbf{x}| = |\lambda| \cdot |\mathbf{x}|$.

A vector whose length is equal to unity is called a *unit vector*. If $\mathbf{x} \in R$ is a nonzero vector, then, as it is easy to see, $\frac{1}{|\mathbf{x}|} \cdot \mathbf{x}$ (the designation $\frac{\mathbf{x}}{|\mathbf{x}|}$ can also be used) is a unit vector.

The inequality $(\mathbf{x}, \mathbf{y})^2 \leqslant (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y})$, known as *Cauchy-Bunyakovsky's inequality*, is valid for any two vectors $\mathbf{x}$ and $\mathbf{y}$ in a Euclidean space.

The equality $(\mathbf{x}, \mathbf{y})^2 = (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y})$ holds if and only if the vectors $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent.

The Cauchy-Bunyakovsky inequality yields $-1 \leqslant \frac{(\mathbf{x}, \mathbf{y})}{|\mathbf{x}| \cdot |\mathbf{y}|} \leqslant 1$. The angle φ specified by the equality $\cos\varphi = \frac{(\mathbf{x}, \mathbf{y})}{|\mathbf{x}| \cdot |\mathbf{y}|}$ and belonging to the closed interval $[0, \pi]$, is said to be the *angle between the vectors* $\mathbf{x}$ *and* $\mathbf{y}$. If $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors, and $\varphi = \pi/2$, then $(\mathbf{x}, \mathbf{y}) = 0$. In this case the vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be *orthogonal* and they are written as $\mathbf{x} \perp \mathbf{y}$.

The following significant relationships hold true for arbitrary vectors $\mathbf{x}$ and $\mathbf{y}$ of a Euclidean space.

1. $|\mathbf{x} + \mathbf{y}| \leqslant |\mathbf{x}| + |\mathbf{y}|$ **(triangle inequality)**.

2. Suppose φ is an angle between the vectors $\mathbf{x}$ and $\mathbf{y}$; then $|\mathbf{x} - \mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2|\mathbf{x}| \cdot |\mathbf{y}| \cos\varphi$ **(law of cosines)**. If $\mathbf{x} \perp \mathbf{y}$, the following equality results: $|\mathbf{x} - \mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2$. Substituting $-\mathbf{y}$ for $\mathbf{y}$ in the last equality, we obtain $|\mathbf{x} + \mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2$ **(theorem of Pythagoras)**.

562. Given the linear space considered in problem 461. Can the scalar product of two arbitrary vectors $\mathbf{x} = (\xi_1; \xi_2; \dots; \xi_n)$ and $\mathbf{y} = (\eta_1; \eta_2; \dots; \eta_n)$ be specified by the equality $(\mathbf{x}, \mathbf{y}) = \xi_1\eta_1 + \xi_2\eta_2 + \dots + \xi_n\eta_n$ (for the space to become Euclidean)?

Solution. Let us verify the fulfilment of conditions 1°-4°.

1°. Since $(\mathbf{y}, \mathbf{x}) = \eta_1\xi_1 + \eta_2\xi_2 + \dots + \eta_n\xi_n$, it follows that $(\mathbf{x}, \mathbf{y}) = (\mathbf{y}, \mathbf{x})$.

2°. Assume $\mathbf{z} = (\zeta_1; \zeta_2; \dots, \zeta_n)$. Then $\mathbf{y} + \mathbf{z} = (\eta_1 + \zeta_1; \eta_2 + \zeta_2; \dots; \eta_n + \zeta_n)$ and

$$(\mathbf{x}, \mathbf{y} + \mathbf{z}) = \xi_1\eta_1 + \xi_1\zeta_1 + \xi_2\eta_2 + \xi_2\zeta_2 + \dots + \xi_n\eta_n + \xi_n\zeta_n$$
$$= (\xi_1\eta_1 + \xi_2\eta_2 + \dots + \xi_n\eta_n) + (\xi_1\zeta_1 + \xi_2\zeta_2 + \dots + \xi_n\zeta_n) = (\mathbf{x}, \mathbf{y}) + (\mathbf{x}, \mathbf{z}).$$

3°. $(\lambda\mathbf{x}, \mathbf{y}) = \lambda\xi_1\eta_1 + \lambda\xi_2\eta_2 + \dots + \lambda\xi_n\eta_n = \lambda(\xi_1\eta_1 + \xi_2\eta_2 + \dots$
$\dots + \xi_n\eta_n) = \lambda(\mathbf{x}, \mathbf{y})$.

4°. $(\mathbf{x}, \mathbf{x}) = \xi_1^2 + \xi_2^2 + \dots + \xi_n^2 \neq 0$ if at least one of the numbers $\xi_1, \xi_2, \dots$
$\dots, \xi_n$ is nonzero.

563. Given the Euclidean space considered in problem 562. Assume $\xi_1, \xi_2, \ldots, \xi_n$ is the quantity of n kinds of goods produced annually by a factory, and $\eta_1, \eta_2, \ldots$ $\ldots, \eta_n$ are the prices of these goods respectively. How can you interpret the scalar product of the vectors $\mathbf{x} = (\xi_1; \xi_2; \ldots; \xi_n)$ and $\mathbf{y} = (\eta_1; \eta_2; \ldots; \eta_n)$?

564. Given the linear space whose vectors are various systems consisting of n positive numbers: $\mathbf{x} = (\xi_1; \xi_2; \ldots; \xi_n)$, $\mathbf{y} = (\eta_1; \eta_2; \ldots; \eta_n)$, $\mathbf{z} = (\zeta_1; \zeta_2; \ldots; \zeta_n)$, $\ldots$. Addition of vectors and multiplication of a vector by a number are specified by the equalities $\mathbf{x} + \mathbf{y} = (\xi_1\eta_1, \xi_2\eta_2, \ldots, \xi_n\eta_n)$, $\lambda\mathbf{x} = (\xi_1^\lambda, \xi_2^\lambda, \ldots, \xi_n^\lambda)$. Can you make this space Euclidean by specifying the scalar product by the equality $(\mathbf{x}, \mathbf{y}) = \ln\xi_1\ln\eta_1 + \ln\xi_2\ln\eta_2 + \ldots + \ln\xi_n\ln\eta_n$?

Solution. Verify the fulfilment of conditions 1°-4°.

1°. $(\mathbf{x}, \mathbf{y}) = \ln\xi_1\ln\eta_1 + \ln\xi_2\ln\eta_2 + \ldots + \ln\xi_n\ln\eta_n$, $(\mathbf{y}, \mathbf{x}) = \ln\eta_1\ln\xi_1 + \ln\eta_2\ln\xi_2 + \ldots + \ln\eta_n\ln\xi_n$, i.e $(\mathbf{x}, \mathbf{y}) = (\mathbf{y}, \mathbf{x})$.

2°. Since $\mathbf{y} + \mathbf{z} = (\eta_1\zeta_1; \eta_2\zeta_2; \eta_n\zeta_n)$ we have

$$(\mathbf{x}, \mathbf{y} + \mathbf{z}) = \ln\xi_1\ln(\eta_1\zeta_1) + \ln\xi_2\ln(\eta_2\zeta_2) + \ldots + \ln\xi_n\ln(\eta_n\zeta_n) =$$
$$= \ln\xi_1\ln\eta_1 + \ln\xi_2\ln\eta_2 + \ldots + \ln\xi_n\ln\eta_n + \ln\xi_1\ln\zeta_1 +$$
$$+ \ln\xi_2\ln\zeta_2 + \ldots + \ln\xi_n\ln\zeta_n = (\mathbf{x}, \mathbf{y}) + (\mathbf{x}, \mathbf{z}).$$

3°. Since $\lambda\mathbf{x} = (\xi_1^\lambda; \xi_2^\lambda; \ldots; \xi_n^\lambda)$ we have

$$(\lambda\mathbf{x}, \mathbf{y}) = \ln\xi_1^\lambda\ln\eta_1 + \ln\xi_2^\lambda\ln\eta_2 + \ldots + \ln\xi_n^\lambda\ln\eta_n =$$
$$= \lambda(\ln\xi_1\ln\eta_1 + \ln\xi_2\ln\eta_2 + \ldots + \ln\xi_n\ln\eta_n) = \lambda(\mathbf{x}, \mathbf{y}).$$

4°. $(\mathbf{x}, \mathbf{x}) = \ln^2\xi_1 + \ln^2\xi_2 + \ldots + \ln^2\xi_n \geqslant 0$.

Consequently the space in question is Euclidean.

565. Consider the linear space of the functions $\mathbf{x} = \mathbf{x}(t)$, $\mathbf{y} = \mathbf{y}(t)$, $\mathbf{z} = \mathbf{z}(t)$, $\ldots$ continuous on the interval $[a, b]$. Can you make the space linear by specifying the scalar product of any two vectors $\mathbf{x}$ and $\mathbf{y}$ by the equality $(\mathbf{x}, \mathbf{y}) = \int_a^b \mathbf{x}(t)\,\mathbf{y}(t)\,dt$?

566. Is the set of all geometric vectors a Euclidean space if the scalar product of two vectors is defined as the product of their lengths?

567. Does the set of all geometric vectors form a Euclidean space if the scalar product of two arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as the product of the length of the vector $\mathbf{a}$ and the trebled projection of the vector $\mathbf{b}$ on the direction of the vector $\mathbf{a}$?

568. Given the linear space, considered in problem 562, for $n = 4$. Determine the angle between the vectors $\mathbf{x} = (4; 1; 2; 2)$ and $\mathbf{y} = (1; 3; 3: -9)$.

Solution. We have

$$|\mathbf{x}| = \sqrt{(\mathbf{x}, \mathbf{x})} = \sqrt{16 + 1 + 4 + 4} = 5;$$
$$|\mathbf{y}| = \sqrt{(\mathbf{y}, \mathbf{y})} = \sqrt{1 + 9 + 9 + 81} = 10;$$
$$(\mathbf{x}, \mathbf{y}) = 4 + 3 + 6 - 18 = -5;$$
$$\cos\varphi = \frac{(\mathbf{x}, \mathbf{y})}{|\mathbf{x}| \cdot |\mathbf{y}|} = \frac{-5}{5 \cdot 10} = -0.1; \quad \varphi = \arccos(-0.1) = 174°15'.$$

569. Given the Euclidean space considered in problem 562. Determine the angle between the vectors $\mathbf{x} = (1; \sqrt{3}; \sqrt{5}, \dots, \sqrt{2n-1})$ and $\mathbf{y} = (1; 0; 0; \dots; 0)$.

570. Consider the Euclidean space of the continuous functions $\mathbf{x}(t), \mathbf{y}(t), \mathbf{z}(t), \dots$ on the closed interval $[-1, 1]$. The scalar product is specified by the equality $(\mathbf{x}, \mathbf{y}) = \int_{-1}^{1} \mathbf{x}(t)\,\mathbf{y}(t)\,dt$. Find the angle between the vectors $\mathbf{x} = 3t^2 - 1$, $\mathbf{y} = 3t - 5t^3$.

Solution. We have

$$(\mathbf{x}, \mathbf{y}) = \int_{-1}^{1} (3t^2 - 1)(3t - 5t^3)\,dt.$$

It is easy to see that $(\mathbf{x}, \mathbf{y}) = 0$ since the integrand function is odd. Consequently, the vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

571. Given the Euclidean space, considered in problem 562, for $n = 6$. Verify the validity of the Pythagorean theorem for the orthogonal vectors $\mathbf{x} = (1; 0; 2; 0; 2; 0)$ and $\mathbf{y} = (0; 6; 0; 3; 0; 2)$.

Solution. We have

$$|\mathbf{x}| = \sqrt{1 + 0 + 4 + 0 + 4 + 0} = 3,$$
$$|\mathbf{y}| = \sqrt{0 + 36 + 0 + 9 + 0 + 4} = 7;$$
$$\mathbf{x} + \mathbf{y} = (1; 6; 2; 3; 2; 2);$$
$$|\mathbf{x} + \mathbf{y}| = \sqrt{1 + 36 + 4 + 9 + 4 + 4} = \sqrt{58}.$$

Thus we have $|\mathbf{x}|^2 + |\mathbf{y}|^2 = |\mathbf{x} + \mathbf{y}|^2$.

572. Two vectors $\mathbf{x} = t^2 + 1$, $\mathbf{y} = \lambda t^2 + 1$ are considered in the Euclidean space of continuous functions corresponding to the condition of the problem 565. Find the value of λ at which the vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal on the closed interval $[0, 1]$ and verify the validity of the Pythagorean theorem for these vectors.

Solution. Set up the scalar product

$$(\mathbf{x}, \mathbf{y}) = \int_0^1 (t^2 + 1)(\lambda t^2 + 1)\,dt = \lambda/5 + (\lambda + 1)/3 + 1.$$

Determining λ from the condition $(\mathbf{x}, \mathbf{y}) = 0$, we have $\lambda/5 + (\lambda + 1)/3 + 1 = 0$, whence $\lambda = -5/2$.

Now find the lengths of the vectors $\mathbf{x} = t^2 + 1$, $\mathbf{y} = -(5/2)t^2 + 1$ and $\mathbf{x} + \mathbf{y} = -(3/2)t^2 + 2$:

$$|\mathbf{x}| = \sqrt{\int_0^1 (t^4 + 2t^2 + 1)\,dt} = \sqrt{\frac{1}{5} + \frac{2}{3} + 1} = \sqrt{\frac{28}{15}},$$

$$|\mathbf{y}| = \sqrt{\int_0^1 \left(\frac{25}{4}t^4 - 5t^2 + 1\right)dt} = \sqrt{\frac{5}{4} - \frac{5}{3} + 1} = \sqrt{\frac{7}{12}},$$

$$|\mathbf{x} + \mathbf{y}| = \sqrt{\int_0^1 \left(\frac{9}{4}t^4 - 6t^2 + 4\right) dt} = \sqrt{\frac{9}{20} - 2 + 4} = \sqrt{\frac{49}{20}}.$$

Thus it follows that $|\mathbf{x}|^2 = 28/15$, $|\mathbf{y}|^2 = 7/12$, $|\mathbf{x} + \mathbf{y}|^2 = 49/20$, i.e. $|\mathbf{x}|^2 + |\mathbf{y}|^2 = |\mathbf{x} + \mathbf{y}|^2$.

573. Consider the set of various ordered systems of the geometric vectors $\mathbf{a}^* = (\mathbf{a}_1; \mathbf{a}_2; \ldots ; \mathbf{a}_n)$, $\mathbf{b}^* = (\mathbf{b}_1; \mathbf{b}_2; \ldots ; \mathbf{b}_n)$, $\mathbf{c}^* = (\mathbf{c}_1; \mathbf{c}_2; \ldots ; \mathbf{c}_n)$, $\ldots$. Is this set Euclidean if addition of elements, multiplication of an element by a number and a scalar product are specified by the equalities $\mathbf{a}^* + \mathbf{b}^* = (\mathbf{a}_1 + \mathbf{b}_1; \mathbf{a}_2 + \mathbf{b}_2; \ldots ; \mathbf{a}_n + \mathbf{b}_n)$, $\lambda\mathbf{a}^* = (\lambda\mathbf{a}_1; \lambda\mathbf{a}_2; \ldots ; \lambda\mathbf{a}_n)$, $(\mathbf{a}^*, \mathbf{b}^*) = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \ldots + \mathbf{a}_n\mathbf{b}_n$ (the right-hand side of the last equality is a sum of the scalar products of the geometric vectors)?

574. Prove the validity of the inequalities

$$\sqrt{(\xi_1 + \eta_1)^2 + (\xi_2 + \eta_2)^2 + \ldots + (\xi_n + \eta_n)^2} \leqslant$$
$$\leqslant \sqrt{\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2} + \sqrt{\eta_1^2 + \eta_2^2 + \ldots + \eta_n^2};$$
$$(\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2)(\eta_1^2 + \eta_2^2 + \ldots + \eta_n^2) \leqslant (\xi_1\eta_1 + \xi_2\eta_2 + \ldots + \xi_n\eta_n)^2,$$

where $\xi_1, \xi_2, \ldots \xi_n, \eta_1, \eta_2, \ldots \eta_n$ are real numbers.

Hint. Make use of the triangle inequality and of the Cauchy-Bunyakovsky inequality for the Euclidean space considered in problem 562.

575. Consider various functions $x(t), y(t), z(t), \ldots$ continuous on the closed interval $[0, 1]$. Prove the validity of the following inequalities:

$$\sqrt{\int_0^1 (x + y)^2\, dt} \leqslant \sqrt{\int_0^1 x^2\, dt} + \sqrt{\int_0^1 y^2\, dt},$$

$$\int_0^1 (y^2/x^2)\, dt \geqslant \left(\int_0^1 y\, dt\right)^2 \Big/ \left(\int_0^1 x^2\, dt\right), \quad \text{if} \quad x(0) \neq 0.$$

5.6. Orthogonal Basis and Orthogonal Transformations

5.6.1. Orthogonal basis. The basis $\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n$ of a Euclidean space is said to be *orthogonal* if $(\mathbf{e}_i, \mathbf{e}_k) = 0$ for $i \neq k$.

The following **theorem** holds true: *there is an orthogonal basis in every Euclidean space.* If an orthogonal basis consists of normalized vectors, the basis is called *orthonormal (normalized orthogonal).* The equalities

$$(\mathbf{e}_i, \mathbf{e}_k) = \begin{cases} 0, & \text{if } i \neq k; \\ 0, & \text{if } i = k \end{cases}$$

are satisfied for the orthogonal basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$.

If some basis $\mathbf{f}_1, \mathbf{f}_2, \dots \mathbf{f}_n$ is known in an n-dimensional Euclidean space, then an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ can always be found in that space.

Any vector $\mathbf{x}$ of a Euclidean space given in an orthonormal basis is specified by the equation

$$\mathbf{x} = \xi_1\mathbf{e}_1 + \xi_2\mathbf{e}_2 + \dots + \xi_n\mathbf{e}_n.$$

The length of the vector $\mathbf{x}$ can be found by the formula

$$|\mathbf{x}| = \sqrt{\xi_1^2 + \xi_2^2 + \dots + \xi_n^2}.$$

Two vectors $\mathbf{x} = \xi_1\mathbf{e}_1 + \xi_2\mathbf{e}_2 + \dots + \xi_n\mathbf{e}_n$ and $\mathbf{y} = \eta_1\mathbf{e}_1 + \eta_2\mathbf{e}_2 + \dots + \eta_n\mathbf{e}_n$ are linearly independent (collinear, proportional) if and only if

$$\xi_1/\eta_1 = \xi_2/\eta_2 = \dots = \xi_n/\eta_n.$$

The condition of orthogonality of the vectors $\mathbf{x}$ and $\mathbf{y}$ has the form

$$\xi_1\eta_1 + \xi_2\eta_2 + \dots + \xi_n\eta_n = 0.$$

The angle between two vectors $\mathbf{x}$ and $\mathbf{y}$ can be found by the formula

$$\cos\varphi = \frac{\xi_1\eta_1 + \xi_2\eta_2 + \dots + \xi_n\eta_n}{\sqrt{\xi_1^2 + \xi_2^2 + \dots + \xi_n^2}\cdot\sqrt{\eta_1^2 + \eta_2^2 + \dots + \eta_n^2}}.$$

In the problems that follow the orthonormal basis of an n-dimensional Euclidean space is designated as $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$.

576. Find the length of the vector $\mathbf{x} = 4\mathbf{e}_1 - 2\mathbf{e}_2 + 2\mathbf{e}_3 - \mathbf{e}_4$.

577. Normalize the vector $\mathbf{x} = \mathbf{e}_1 + 2\sqrt{2}\mathbf{e}_2 + 3\sqrt{3}\mathbf{e}_3 + 8\mathbf{e}_4 + 5\sqrt{5}\mathbf{e}_5$.

578. Given the matrix

$$A = \begin{pmatrix} -2/7 & 3/7 & 6/7 \\ 6/7 & -2/7 & 3/7 \\ 3/7 & 6/7 & -2/7 \end{pmatrix}$$

of transition from the orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to the basis $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$. Prove that the basis $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ is orthonormal.

579. Normalize the vector $\mathbf{x} = \mathbf{e}_1 \sin^3\alpha + \mathbf{e}_2 \sin^2\alpha \cos\alpha + \mathbf{e}_3 \sin\alpha \cos\alpha + \mathbf{e}_4 \cos\alpha$.

580. Determine the angle between the vectors $\mathbf{x} = \mathbf{e}_1\sqrt{7} + \mathbf{e}_2\sqrt{5} + \mathbf{e}_3\sqrt{3} + \mathbf{e}_4$ and $\mathbf{y} = \mathbf{e}_1\sqrt{7} + \mathbf{e}_2\sqrt{5}$.

581. Find the unit vector orthogonal to the vectors $\mathbf{x} = 3\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4$, $\mathbf{y} = \mathbf{e}_1 - 3\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4$, $\mathbf{z} = \mathbf{e}_1 + \mathbf{e}_2 - 3\mathbf{e}_3 + \mathbf{e}_4$.

582. At what value of λ are the vectors $\mathbf{x} = \lambda\mathbf{e}_1 + \lambda\mathbf{e}_2 - \mathbf{e}_3 - \lambda\mathbf{e}_4$ and $\mathbf{y} = \mathbf{e}_1 - \mathbf{e}_2 + \lambda\mathbf{e}_3 - \mathbf{e}_4$ of the same length?

583. A basis $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4$ is given in a 4-dimensional space. Use the vectors of this basis to construct the orthonormal basis of the same space.

Solution. First construct in the given space an arbitrary basis $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4$.

Put $\mathbf{g}_1 = \mathbf{f}_1$, $\mathbf{g}_2 = \mathbf{f}_2 + \alpha\mathbf{g}_1$. Select real numbers α such that the condition $\mathbf{g}_2 \perp \mathbf{g}_1$ is complied with. Scalar multiplication of both sides of the last equation by $\mathbf{g}_1$ results in

$$(\mathbf{g}_1, \mathbf{g}_2) = (\mathbf{g}_1, \mathbf{f}_2) + \alpha(\mathbf{g}_1, \mathbf{g}_1).$$

Since $(\mathbf{g}_1, \mathbf{g}_2) = 0$, it follows that $\alpha = -(\mathbf{g}_1, \mathbf{f}_2)/(\mathbf{g}_1, \mathbf{g}_1)$.

Next, in the equality $\mathbf{g}_3 = \mathbf{f}_3 + \beta_1\mathbf{g}_1 + \beta_2\mathbf{g}_2$ choose β_1 and β_2 such that the conditions $\mathbf{g}_3 \perp \mathbf{g}_1$, $\mathbf{g}_3 \perp \mathbf{g}_2$ are fulfilled. From the equalities

$$(\mathbf{g}_1, \mathbf{g}_3) = (\mathbf{g}_1, \mathbf{f}_3) + \beta_1(\mathbf{g}_1, \mathbf{g}_1) + \beta_2(\mathbf{g}_1, \mathbf{g}_2),$$
$$(\mathbf{g}_2, \mathbf{g}_3) = (\mathbf{g}_2, \mathbf{f}_3) + \beta_1(\mathbf{g}_1, \mathbf{g}_2) + \beta_2(\mathbf{g}_2, \mathbf{g}_2)$$

we obtain $\beta_1 = -(\mathbf{g}_1, \mathbf{f}_3)/(\mathbf{g}_1, \mathbf{g}_1)$, $\beta_2 = -(\mathbf{g}_2, \mathbf{f}_3)/(\mathbf{g}_2, \mathbf{g}_2)$.

Finally, from the equality $\mathbf{g}_4 = \mathbf{f}_4 + \gamma_1\mathbf{g}_1 + \gamma_2\mathbf{g}_2 + \gamma_3\mathbf{g}_3$ we find $\gamma_1 = -(\mathbf{g}_1, \mathbf{f}_4)/(\mathbf{g}_1, \mathbf{g}_1)$, $\gamma_2 = -(\mathbf{g}_2, \mathbf{f}_4)/(\mathbf{g}_2, \mathbf{g}_2)$, $\gamma_3 = -(\mathbf{g}_3, \mathbf{f}_4)/(\mathbf{g}_3, \mathbf{g}_3)$.

Thus we see that with the choice of $\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3$ having been made, the vectors $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4$ are pairwise orthogonal. Hence, the vectors $\mathbf{e}_1 = \mathbf{g}_1/|\mathbf{g}_1|$, $\mathbf{e}_2 = \mathbf{g}_2/|\mathbf{g}_2|$, $\mathbf{e}_3 = \mathbf{g}_3/|\mathbf{g}_3|$, $\mathbf{e}_4 = \mathbf{g}_4/|\mathbf{g}_4|$ form an orthonormal basis.

584. Consider a Euclidean space of polynomials of degree not higher than the second. The scalar product of two arbitrary polynomials $\mathbf{x} = \mathbf{x}(t)$ and $\mathbf{y} = \mathbf{y}(t)$ is specified by the equality $(\mathbf{x}, \mathbf{y}) = \int_0^1 \mathbf{x}(t)\mathbf{y}(t)\,dt$. Using the basis $\mathbf{f}_1 = t^2$, $\mathbf{f}_2 = t$, $\mathbf{f}_3 = 1$, and employing the method of solutions considered in problem 583, construct for this space an orthonormal basis.

Solution. First construct the orthogonal basis $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$. Put $\mathbf{g}_1 = \mathbf{f}_1$, i.e. $\mathbf{g}_1 = t^2$, $\mathbf{g}_2 = \mathbf{f}_2 + \alpha\mathbf{g}_1 = t + \alpha t^2$. Then

$$\int_0^1 \mathbf{g}_2 t^2 dt = \int_0^1 t^3 dt + \alpha \int_0^1 t^4 dt.$$

Because of the orthogonality of the vectors $\mathbf{g}_1$ and $\mathbf{g}_2$, the left-hand side of the last equality vanishes. Thus, $\alpha = -5/4$ and $\mathbf{g}_2 = t - 5t^2/4$.

Now find $\mathbf{g}_3$. In the equation $\mathbf{g}_3 = 1 + \beta_1 t + \beta_2(t - 5t^2/4)$ the values of β_1 and β_2 can be found from the conditions of orthogonality

$$\int_0^1 \mathbf{g}_3 t^2 dt = 0 \quad \text{and} \quad \int_0^1 \mathbf{g}_3\left(t - \frac{5}{4}t^2\right) dt = 0.$$

It follows that

$$0 = \int_0^1 t^2 dt + \beta_1 \int_0^1 t^4 dt$$

and

$$0 = \int_0^1 \left(t - \frac{5}{4}t^2\right) dt + \beta_2 \int_0^1 \left(t - \frac{5}{4}t^2\right) dt.$$

Hence $\beta_1 = -5/3$, $\beta_2 = -4$ and $g_3 = 1 - 5t^2/3 - 4(t - 5t^2/4)$, i.e. $g_3 = 1 - 4t + 10t^2/3$.

Find the lengths of the vectors $g_1 = t^2$, $g_2 = t - 5t^2/4$ and $g_3 = 1 - 4t + 10t^2/3$:

$$|g_1| = \sqrt{\int_0^1 t^4 dt} = \frac{1}{\sqrt{5}},$$

$$|g_2| = \sqrt{\int_0^1 \left(t - \frac{5}{4}t^2\right)^2 dt} = \sqrt{\frac{1}{3} - \frac{5}{8} + \frac{5}{16}} = \frac{1}{4\sqrt{3}},$$

$$|g_3| = \sqrt{\int_0^1 \left(1 - 4t + \frac{10}{3}t^2\right)^2 dt} = \sqrt{\int_0^1 \left(1 - 8t + \frac{68}{3}t^2 - \frac{80}{3}t^3 + \frac{100}{9}t^4\right) dt}$$

$$= \sqrt{1 - 4 + \frac{68}{3} - \frac{20}{3} + \frac{20}{9}} = \frac{1}{3}.$$

Thus we see that the vectors

$$e_1 = g_1/|g_1| = \sqrt{5}t^2, \quad e_2 = g_2/|g_2| = \sqrt{3}(4 - 5t^2),$$
$$e_3 = g_3/|g_3| = 3 - 12t + 10t^2$$

form an orthonormal basis.

585. At what value of λ is the basis formed by the vectors $g_1 = \lambda e_1 + e_2 + e_3 + e_4$, $g_2 = e_1 + \lambda e_2 + e_3 + e_4$, $g_3 = e_1 + e_2 + \lambda e_3 + e_4$, $g_4 = e_1 + e_2 + e_3 + \lambda e_4$ orthogonal? Normalize this basis.

Solution. From the condition $(e_i, e_k) = 0$ (at $i \neq k$), we obtain the equation $\lambda + \lambda + 1 + 1 = 0$. Consequently, $\lambda = -1$ and $g_1 = -e_1 + e_2 + e_3 + e_4$, $g_2 = e_1 - e_2 + e_3 + e_4$, $g_3 = e_1 + e_2 - e_3 + e_4$, $g_4 = e_1 + e_2 + e_3 - e_4$, $|g_i| = \sqrt{1 + 1 + 1 + 1} = 2$. Thus the vectors $e'_1 = 0.5(-e_1 + e_2 + e_3 + e_4)$, $e'_2 = 0.5(e_1 - e_2 + e_3 + e_4)$, $e'_3 = 0.5(e_1 + e_2 - e_3 + e_4)$, $e'_4 = 0.5(e_1 + e_2 + e_3 - e_4)$ form an orthonormal basis.

586. At what values of α and β is the basis formed by the vectors $e'_1 = \frac{\alpha}{3}e_1 + \frac{1-\alpha}{3}e_2 + \beta e_3$, $e'_2 = \frac{1-\alpha}{3}e_1 + \beta e_2 + \frac{\alpha}{3}e_3$, $e'_3 = \beta e_1 + \frac{\alpha}{3}e_2 + \frac{1-\alpha}{3}e_3$ orthonormal?

Solution. From the conditions $|\mathbf{e}'_i| = 1$, $(\mathbf{e}'_i, \mathbf{e}'_k) = 0$ (at $i \neq k$), we obtain a system of equations

$$\begin{cases} \alpha^2 + (1 - \alpha)^2 + 9\beta^2 = 9, \\ \alpha(1 - \alpha) + 3(1 - \alpha)\beta + 3\alpha\beta = 0. \end{cases}$$

From the last equation we find $\beta = -\alpha(\alpha - 1)/3$. Substituting this value of β in the first equation, we find

$$\alpha^2 + (1 - \alpha)^2 + \alpha^2(1 - \alpha^2)^2 = 9;\ 1 - 2(1 - \alpha)\alpha + \alpha^2(1 - \alpha)^2 = 9;$$
$$(1 - \alpha + \alpha^2)^2 = 9.$$

Since $1 - a + \alpha^2 > 0$ at real values of α, it follows that $1 - \alpha + \alpha^2 = 3$, i.e. $\alpha^2 - \alpha - 2 = 0$. Consequently, $\alpha_1 = -1$, $\alpha_2 = 2$, $\beta_1 = -2/3$, $\beta_2 = 2/3$.

Thus we obtain two orthonormal bases:

$$\mathbf{e}_1^{(1)} = -\frac{1}{3}\mathbf{e}_1 + \frac{2}{3}\mathbf{e}_2 - \frac{2}{3}\mathbf{e}_3,\quad \mathbf{e}_2^{(1)} = \frac{2}{3}\mathbf{e}_1 - \frac{2}{3}\mathbf{e}_2 - \frac{1}{3}\mathbf{e}_3,$$
$$\mathbf{e}_3^{(1)} = -\frac{2}{3}\mathbf{e}_1 - \frac{1}{3}\mathbf{e}_2 + \frac{2}{3}\mathbf{e}_3,\quad \mathbf{e}_1^{(2)} = \frac{2}{3}\mathbf{e}_1 - \frac{1}{3}\mathbf{e}_2 + \frac{2}{3}\mathbf{e}_3,$$
$$\mathbf{e}_2^{(2)} = -\frac{1}{3}\mathbf{e}_1 + \frac{2}{3}\mathbf{e}_2 + \frac{2}{3}\mathbf{e}_3,\quad \mathbf{e}_3^{(2)} = \frac{2}{3}\mathbf{e}_1 + \frac{2}{3}\mathbf{e}_2 - \frac{1}{3}\mathbf{e}_3.$$

5.6.2. Orthogonal transformations. A linear transformation A is said to be *orthogonal* if it does not bring about any change in the length of any vector of a Euclidean space, i.e. $|\mathbf{Ax}| = |\mathbf{x}|$. In this case the orthogonal transformation A does not change the scalar product of any two vectors $\mathbf{x}$ and $\mathbf{y}$ of a Euclidean space, i.e. $(\mathbf{Ax}, \mathbf{Ay}) = (\mathbf{x}, \mathbf{y})$. Thus,

$$\frac{(\mathbf{x}, \mathbf{y})}{|\mathbf{x}| \cdot |\mathbf{y}|} = \frac{(\mathbf{Ax}, \mathbf{Ay})}{|\mathbf{Ax}| \cdot |\mathbf{Ay}|}.$$

It follows from the last equality that the orthogonal transformation A does not change the angle between any two vectors $\mathbf{x}$ and $\mathbf{y}$.

An orthogonal transformation carries any orthonormal basis into an orthonormal basis. Conversely, if a linear transformation carries any orthonormal basis into an orthonormal basis, then it is orthogonal.

587. Is a transformation, which carries each geometric vector into a vector symmetric about some fixed plane, orthogonal?

588. Is a transformation, which consists in a rotation of any vector lying on the plane xOy through a fixed angle α, orthogonal?

589. At what values of λ is the transformation A, specified by the equality $\mathbf{Ax} = \lambda\mathbf{x}$, orthogonal?

590. Consider the transformation A, determined in some orthonormal basis $\mathbf{e}_1$,

e_2, e_3 by the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Is it orthogonal if

$$a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0, \quad a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = 0,$$
$$a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0,$$
$$a_{11}^2 + a_{21}^2 + a_{31}^2 = 1, \quad a_{12}^2 + a_{22}^2 + a_{32}^2 = 1, \quad a_{13}^2 + a_{23}^2 + a_{33}^2 = 1?$$

591. Consider the transformation $A\mathbf{x} = -\xi_1\mathbf{e}_1 + \xi_2\mathbf{e}_2 + \xi_3\mathbf{e}_3 + \xi_4\mathbf{e}_4$, where $\mathbf{x} = \xi_1\mathbf{e}_1 + \xi_2\mathbf{e}_2 + \xi_3\mathbf{e}_3 + \xi_4\mathbf{e}_4$ is an arbitrary vector and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$, an orthonormal basis. Is it orthogonal?

592. Suppose $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6$ is an orthonormal basis. Prove that A is an orthonormal transformation if $A\mathbf{e}_1 = \mathbf{e}_1$, $A\mathbf{e}_2 = -\mathbf{e}_2$, $A\mathbf{e}_3 = \mathbf{e}_3\cos\alpha + \mathbf{e}_4\sin\alpha$, $A\mathbf{e}_4 = -\mathbf{e}_3\sin\alpha + \mathbf{e}_4\cos\alpha$, $A\mathbf{e}_5 = \mathbf{e}_5\cos\beta + \mathbf{e}_6\sin\beta$, $A\mathbf{e}_6 = -\mathbf{e}_5\sin\beta + \mathbf{e}_6\cos\beta$.

5.7. Quadratic Forms

A *quadratic form* of the real variables $x_1, x_2, \ldots, x_n$ is a polynomial of the second degree in these variables which does not contain a constant term and terms of the first degree.

If $f(x_1, x_2, \ldots, x_n)$ is a quadratic form of the variables $x_1, x_2, \ldots, x_n$ and λ is some real number, then $f(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^2 f(x_1, x_2, \ldots, x_n)$.

If $n = 2$, then

$$f(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2.$$

If $n = 3$, then

$$f(x_1, x_2, x_3) = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3.$$

In what follows we present all the necessary definitions and formulas for the case of a quadratic form of three variables.

The matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

in which $a_{ik} = a_{ki}$ is called a *matrix of the quadratic form* $f(x_1, x_2, x_3)$, and the corresponding determinant is known as the *determinant of this quadratic form*.

Since A is a symmetric matrix, the roots λ_1, λ_2 and λ_3 of the characteristic equation

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$$

are real numbers.

Assume that

$$\mathbf{e}'_1 = b_{11}\mathbf{e}_1 + b_{21}\mathbf{e}_2 + b_{31}\mathbf{e}_3,$$
$$\mathbf{e}'_2 = b_{12}\mathbf{e}_1 + b_{22}\mathbf{e}_2 + b_{32}\mathbf{e}_3,$$
$$\mathbf{e}'_3 = b_{13}\mathbf{e}_1 + b_{23}\mathbf{e}_2 + b_{33}\mathbf{e}_3$$

are unit eigenvectors corresponding to the characteristic numbers λ_1, λ_2, λ_3 in the orthonormal basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. In their turn, the vectors $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$ form an orthonormal basis. The matrix

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

is the transition matrix from the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ to the basis $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$.

The formulas of transformation of coordinates upon transition to a new orthonormal basis have the form

$$x_1 = b_{11}x'_1 + b_{12}x'_2 + b_{13}x'_3,$$
$$x_2 = b_{21}x'_1 + b_{22}x'_2 + b_{23}x'_3,$$
$$x_3 = b_{31}x'_1 + b_{32}x'_2 + b_{23}x'_3.$$

Using these formulas to transform the quadratic form $f(x_1, x_2, x_3)$, we obtain the quadratic form

$$f(x'_1, x'_2, x'_3) = \lambda_1 x'^2_1 + \lambda_2 x'^2_2 + \lambda_3 x'^2_3,$$

which does not contain terms with the products $x'_1x'_2$, $x'_1x'_3$, $x'_2x'_3$.

It is customary to say that the quadratic form $f(x_1, x_2, x_3)$ has been reduced to the *canonical form* by means of the orthogonal transformation B. The argument was carried out under the assumption that the characteristic numbers λ_1, λ_2, λ_3 were distinct. In the problems we will show what to do in the case when there are identical characteristic numbers.

593. Reduce the quadratic form

$$f = 27x_1^2 - 10x_1x_2 + 3x_2^2$$

to the canonical form.

Solution. Here $a_{11} = 27$, $a_{12} = -5$, $a_{22} = 3$. Derive a characteristic equation

$$\begin{vmatrix} 27-\lambda & -5 \\ -5 & 3-\lambda \end{vmatrix} = 0, \quad \text{or} \quad \lambda^2 - 30\lambda + 56 = 0,$$

that is, the characteristic numbers $\lambda_1 = 2$, $\lambda_2 = 28$.

Determine the eigenvectors. If $\lambda = 2$, we obtain a system of equations

$$\begin{cases} 25\xi_1 - 5\xi_2 = 0, \\ -5\xi_1 + \xi_2 = 0. \end{cases}$$

Thus we have $\xi_2 = 5\xi_1$. Putting $\xi_1 = c$, we obtain $\xi_2 = 5c$, that is, the eigenvector $\mathbf{u} = c(\mathbf{e}_1 + 5\mathbf{e}_2)$.

If $\lambda = 28$, we arrive at a system

$$\begin{cases} -\xi_1 - 5\xi_2 = 0, \\ -5\xi_1 - 25\xi_2 = 0. \end{cases}$$

In this case we obtain an eigenvector $\mathbf{v} = c(-5\mathbf{e}_1 + \mathbf{e}_2)$.

To normalize the vectors u and v, we must assume $c = 1/\sqrt{1^2 + 5^2} = 1/\sqrt{26}$. Thus we have found the normalized eigenvectors $\mathbf{e}_1' = (\mathbf{e}_1 + 5\mathbf{e}_2)/\sqrt{26}$, $\mathbf{e}_2' = (-5\mathbf{e}_1 + \mathbf{e}_2)/\sqrt{26}$.

The transition matrix from the orthonormal basis $\mathbf{e}_1, \mathbf{e}_2$ to the orthonormal basis $\mathbf{e}_1', \mathbf{e}_2'$ has the form

$$B = \begin{pmatrix} 1/\sqrt{26} & -5/\sqrt{26} \\ 5/\sqrt{26} & 1/\sqrt{26} \end{pmatrix}.$$

Hence the formulas for coordinate transformation:

$$x_1 = \frac{1}{\sqrt{26}} x_1' - \frac{5}{\sqrt{26}} x_2',$$

$$x_2 = \frac{5}{\sqrt{26}} x_1' + \frac{1}{\sqrt{26}} x_2'.$$

It follows that

$$f = 27\left(\frac{1}{\sqrt{26}}x_1' - \frac{5}{\sqrt{26}}x_2'\right)^2 - 10\left(\frac{1}{\sqrt{26}}x_1' - \frac{5}{\sqrt{26}}x_2'\right)\left(\frac{5}{\sqrt{26}}x_1' + \frac{1}{\sqrt{26}}x_2'\right) + 3\left(\frac{5}{\sqrt{26}}x_1' + \frac{1}{\sqrt{26}}x_2'\right)^2 = 2x_1'^2 + 28x_2'^2.$$

This result could be obtained directly since $f = \lambda_1 x_1'^2 + \lambda_2 x_2'^2$.

594. Reduce the quadratic form

$$f = 2x_1^2 + 8x_1x_2 + 8x_2^2$$

to the canonical form.

Solution. Here $a_{11} = 2$, $a_{12} = 4$, $a_{22} = 8$. We solve the characteristic equation

$$\begin{vmatrix} 2-\lambda & 4 \\ 4 & 8-\lambda \end{vmatrix} = 0, \quad \lambda_1 = 0, \quad \lambda_2 = 10.$$

Next we determine the eigenvectors. At $\lambda = 0$, we obtain a system

$$\begin{cases} 2\xi_1 + 4\xi_2 = 0, \\ 4\xi_1 + 8\xi_2 = 0, \end{cases}$$

which has a solution $\xi_1 = 2c$, $\xi_2 = -c$, i.e. $\mathbf{u} = c(2\mathbf{e}_1 - \mathbf{e}_2)$.

At $\lambda = 10$ we have

$$\begin{cases} -8\xi_1 + 4\xi_2 = 0, \\ 4\xi_1 - 2\xi_2 = 0, \end{cases}$$

whence $\xi_1 = c$, $\xi_2 = 2c$, i.e. $\mathbf{v} = c(\mathbf{e}_1 + 2\mathbf{e}_2)$.

Assuming $c = 1/\sqrt{2^2 + 1^2} = 1/\sqrt{5}$, we find the normalized eigenvectors $\mathbf{e}_1' = (2\mathbf{e}_1 - \mathbf{e}_2)/\sqrt{5}$, $\mathbf{e}_2' = (\mathbf{e}_1 + 2\mathbf{e}_2)/\sqrt{5}$.

The matrix of transition to a new basis (orthogonal transformation matrix) has the form

$$B = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}.$$

The formulas for transformation of coordinates will be written as

$$x_1 = \frac{2}{\sqrt{5}}x_1' + \frac{1}{\sqrt{5}}x_2',$$
$$x_2 = -\frac{1}{\sqrt{5}}x_1' + \frac{2}{\sqrt{5}}x_2'.$$

Consequently,

$$f = 2\left(\frac{1}{\sqrt{5}}x_1' + \frac{1}{\sqrt{5}}x_2'\right)^2 + 8\left(\frac{2}{\sqrt{5}}x_1' + \frac{1}{\sqrt{5}}x_2'\right)\left(-\frac{1}{\sqrt{5}}x_1' + \frac{2}{\sqrt{5}}x_2'\right) + $$
$$+ 8\left(-\frac{1}{\sqrt{5}}x_1' + \frac{2}{\sqrt{5}}x_2'\right)^2 = 10x_2'^2.$$

The problem can be solved in a simpler way. Note that $f = 2(x_1 + 2x_2)^2$; therefore, we can assume $x_2' = (x_1 + 2x_2)/\sqrt{1+4} = (x_1 + 2x_2)/\sqrt{5}$, $x_1' = (2x_1 - x_2)/\sqrt{5}$ (the second equality has been written with due account of the orthogonality of the transformation). Since $x_1 + 2x_2 = \sqrt{5}x_2'$, it follows that $f = 10x_2'^2$.

595. Reduce the quadratic form

$$f = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$$

to the canonical form.

Solution. Here $a_{11} = 3, a_{22} = 2, a_{33} = 1, a_{12} = 2, a_{13} = 0, a_{23} = 2$. We derive a characteristic equation

$$\begin{vmatrix} 3-\lambda & 2 & 0 \\ 2 & 2-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{vmatrix} = 0; \quad (3-\lambda)(2-\lambda)(1-\lambda) -$$

$$- 4(1-\lambda) + 4(3-\lambda) = 0;$$

$$(3-\lambda)(2-\lambda)(1-\lambda) - 8(2-\lambda) = 0; \quad (2-\lambda)(\lambda^2 - 4\lambda + 3 - 8) = 0;$$

$$(2-\lambda)(\lambda^2 - 4\lambda - 5) = 0; \quad \lambda_1 = 2, \quad \lambda_2 = -1, \quad \lambda_3 = 5.$$

Now we determine the eigenvectors corresponding to the characteristic numbers we have found. To determine the coordinates of the eigenvectors, we obtain three systems of linear equations:

$$(1)\ \lambda = 2, \qquad (2)\ \lambda = -1, \qquad (3)\ \lambda = 5,$$

$$\begin{cases} \xi_1 + 2\xi_2 = 0, \\ 2\xi_1 + 2\xi_3 = 0, \\ 2\xi_2 - \xi_3 = 0; \end{cases} \begin{cases} 4\xi_1 + 2\xi_2 = 0, \\ 2\xi_1 + 3\xi_2 + 2\xi_3 = 0, \\ 2\xi_2 + 2\xi_3 = 0; \end{cases} \begin{cases} -2\xi_1 + 2\xi_2 = 0, \\ 2\xi_1 - 3\xi_2 + 2\xi_3 = 0, \\ 2\xi_2 - 4\xi_3 = 0; \end{cases}$$

$$\xi_1 = 2c, \quad \xi_2 = -c, \quad \xi_3 = -2c, \quad \xi_1 = c, \quad \xi_2 = -2c, \quad \xi_3 = 2c,$$

$$\xi_1 = 2c, \quad \xi_2 = 2c, \quad \xi_3 = c,$$

$$\mathbf{u} = c(2\mathbf{e}_1 - \mathbf{e}_2 - 2\mathbf{e}_3), \quad \mathbf{v} = c(\mathbf{e}_1 - 2\mathbf{e}_2 + 2\mathbf{e}_3), \quad \mathbf{w} = c(2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3),$$

$$\mathbf{e}_1' = \frac{1}{3}(2\mathbf{e}_1 - \mathbf{e}_2 - 2\mathbf{e}_3); \quad \mathbf{e}_2' = \frac{1}{3}(\mathbf{e}_1 - 2\mathbf{e}_2 + 2\mathbf{e}_3); \quad \mathbf{e}_3' = \frac{1}{3}(2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3).$$

The orthogonal transformation matrix has the form

$$B = \begin{pmatrix} 2/3 & 1/3 & 2/3 \\ -1/3 & -2/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{pmatrix}.$$

The formulas for the transformation of coordinates are

$$x_1 = \frac{2}{3}x_1' + \frac{1}{3}x_2' + \frac{2}{3}x_3', \quad x_2 = -\frac{1}{3}x_1' - \frac{2}{3}x_2' + \frac{2}{3}x_3',$$

$$x_3 = -\frac{2}{3}x_1' + \frac{2}{3}x_2' + \frac{1}{3}x_3'.$$

It follows that $f = 2x_1'^2 - x_2'^2 + 5x_3'^2$.

596. Reduce the following quadratic form to the canonical form:

$f = 6x_1^2 + 3x_2^2 + 3x_3^2 + 4x_1x_2 + 4x_1x_3 - 8x_2x_3$.

Solution. Here $a_{11} = 6$, $a_{22} = 3$, $a_{33} = 3$, $a_{12} = 2$, $a_{13} = 2$, $a_{23} = -4$. Solving the characteristic equation

$$\begin{vmatrix} 6-\lambda & 2 & 2 \\ 2 & 3-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0,$$

we find the characteristic numbers $\lambda_1 = \lambda_2 = 7$, $\lambda_3 = -2$.

At $\lambda = 7$, we arrive at a system

$$\begin{cases} -\xi_1 + 2\xi_2 + 2\xi_3 = 0, \\ 2\xi_1 - 4\xi_2 - 4\xi_3 = 0, \\ 2\xi_1 - 4\xi_2 - 4\xi_3 = 0, \end{cases}$$

which can be reduced to one equation $\xi_1 = 2\xi_2 + 2\xi_3$. The solution of this system can be written in the form $\xi_1 = 2a + 2b$, $\xi_2 = a$, $\xi_3 = b$. As a result we obtain a family of eigenvectors $\mathbf{u} = 2(a + b)\mathbf{e}_1 + a\mathbf{e}_2 + b\mathbf{e}_3$, dependent on two parameters a and b.

At $\lambda = -2$ we obtain a system

$$\begin{cases} 8\xi_1 + 2\xi_2 + 2\xi_3 = 0, \\ 2\xi_1 + 5\xi_2 - 4\xi_3 = 0, \\ 2\xi_1 - 4\xi_2 + 5\xi_3 = 0. \end{cases}$$

Solving the two last equations, for instance, we obtain

$$\xi_1/9 = \xi_2/(-18) = \xi_3/(-18) \quad \text{or}$$

$$\xi_1 = -\xi_2/2 = -\xi_3/2; \quad \xi_1 = c, \quad \xi_2 = -2c, \quad \xi_3 = -2c.$$

Thus we have a one-parameter family of eigenvectors $\mathbf{v} = c(\mathbf{e}_1 - 2\mathbf{e}_2 - 2\mathbf{e}_3)$.

Let us isolate, from the family of the eigenvectors $\mathbf{u} = 2(a + b)\mathbf{e}_1 + a\mathbf{e}_2 + b\mathbf{e}_3$, any two orthogonal vectors. Putting $a = 0$, $b = 1$, for example, we obtain an eigenvector $\mathbf{u}_1 = 2\mathbf{e}_1 + \mathbf{e}_3$. We choose the parameters a and b such that the equality $(\mathbf{u}, \mathbf{u}_1) = 0$ is satisfied. Then we obtain an equation $2 \cdot 2(a + b) + b = 0$, i.e. $4a + 5b = 0$. Now we can assume $a = 5$, $b = -4$; this yields the other eigenvector of the family in question: $\mathbf{u}_2 = 2\mathbf{e}_1 + 5\mathbf{e}_2 - 4\mathbf{e}_3$.

Thus we have obtained three pairwise orhogonal vectors: $\mathbf{u}_1 = 2\mathbf{e}_1 + \mathbf{e}_3$, $\mathbf{u}_2 = 2\mathbf{e}_1 + 5\mathbf{e}_2 - 4\mathbf{e}_3$, $\mathbf{v} = \mathbf{e}_1 - 2\mathbf{e}_2 - 2\mathbf{e}_3$. The eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ correspond to the characteristic number $\lambda = 7$, and the eigenvector $\mathbf{v}$, to the characteristic number $\lambda = -2$ at $c = 1$.

Normalizing these vectors, we receive a new orthonormal basis, the transition matrix to the new basis being of the form

$$B = \begin{pmatrix} 2/\sqrt{5} & 2/(3\sqrt{5}) & 1/3 \\ 0 & \sqrt{5}/3 & -2/3 \\ 1/\sqrt{5} & -4/(3\sqrt{5}) & -2/3 \end{pmatrix}.$$

Applying the formulas for the transformation of coordinates

$$x_1 = \frac{2}{\sqrt{5}}x_1' + \frac{2}{3\sqrt{5}}x_2' + \frac{1}{3}x_3', \quad x_2 = \frac{\sqrt{5}}{3}x_2' - \frac{2}{3}x_3',$$

$$x_3 = \frac{1}{\sqrt{5}}x_1' - \frac{4}{3\sqrt{5}}x_2' - \frac{2}{3}x_3'$$

to the given quadratic form, we get $f = 7x_1'^2 + 7x_2'^2 - 2x_3'^2$.

Chapter 6

Introduction to Analysis

6.1. Absolute and Relative Errors

Suppose a is a number approximating in calculations the exact number A. The *absolute error* of the approximate number a is the absolute value of the difference between this number and the corresponding exact number: $|A - a|$.

The *limit of the absolute error* is the smallest number Δ possible which satisfies the inequality $|A - a| \leqslant \Delta$.

The exact number A lies between the boundaries $a - \Delta \leqslant A \leqslant a + \Delta$, or $A = a \pm \Delta$.

The *relative error* of the number a is the ratio between the absolute error of that number and the corresponding exact number: $|A - a|/A$.

The *limit of the relative error* is the smallest number δ possible which satisfies the inequality $|A - a|/A \leqslant \delta$.

Since in practical calculations $A \approx a$, the number $\delta = \Delta/a$ (usually expressed in per cent) is taken as the limit of the relative error.

The inequality $a(1 - \delta) \leqslant A \leqslant a(1 + \delta)$ holds true.

An approximate positive number a written as a decimal fraction is said to have n correct decimal digits if the absolute error of that number does not exceed half the unity in the nth decimal place.

At $n > 1$, the number $\delta = \frac{1}{2k}\left(\frac{1}{10}\right)^{n-1}$ can be taken as the limit of the relative error of the approximate number a with the first significant digit k.

If it is known that

$$\delta \leqslant \frac{1}{2(k+1)}\left(\frac{1}{10}\right)^{n-1}, \tag{1}$$

then the number a has n correct digits.

The limit of the absolute error of the algebraic sum of several numbers is equal to the sum of the limits of the absolute errors of the summands.

The relative error of the sum of positive summands does not exceed the greatest relative error of the summands.

The limit of the relative error of the product and the quotient of approximate numbers is equal to the sum of the limits of the relative errors of those numbers.

The limit of the relative error of the degree of an approximate number is equal to the product of the limit of the relative error of that number by the exponent.

597. The angle measured by a theodolite turned out to be equal to $22°20'30'' \pm 30''$. What is the relative error of the measurement?

Solution. The absolute error $\Delta = 30''$. Then the relative error

$$\delta = \frac{\Delta}{a} = \frac{30''}{22°20'30''} \cdot 100\% = 0.04\%.$$

598. Determine the number of the correct digits and write the corresponding approximate value of the acceleration due to gravity $g = 9.806\ldots$, the relative error being 0.5%.

Solution. We use inequality (1) and obtain $0.005 \leqslant \frac{1}{2 \cdot 10}\left(\frac{1}{10}\right)^{n-1}$ since the first significant digit is 9, i.e. $n = 2$. Hence $g = 9.8$.

599. The limit of the relative error of the number $\sqrt{19}$ is known to be 0.1%. How many correct digits are there in the number?

Solution. Here the first significant digit is 4, the limit of the relative error $\delta = 0.001 = 10^{-3}$. From inequality (1) we have $0.001 \leqslant \frac{1}{2 \cdot 5}\left(\frac{1}{10}\right)^{n-1}$, whence $n = 3$. Consequently, $\sqrt{19} = 4.36$ ($\sqrt{19} = 4.3589$ by the 4-digit tables).

600. How many correct digits are there in the number $A = 3.7563$ if the relative error is 1%?

Solution. The first correct digit is 3; therefore, $0.01 \leqslant \frac{1}{2 \cdot 4}\left(\frac{1}{10}\right)^{n-1}$, whence $n = 2$. The number A should be written as $A = 3.8$.

601. The area of a square is equal to 25.16 sq. cm (with an accuracy to within 0.01 sq. cm). Determine the relative error and the number of correct digits in measuring the side of the square.

Solution. The sought-for side $x = \sqrt{25 \cdot 16}$. The relative error of the side of the square $\delta = (1/2) \cdot (0.01/25.16)$, where 0.01 is the absolute error of the area, i.e. $\delta = 0.0002$. The first significant digit of the number measuring the side of the square is 5. Solving inequality (1) at $k = 5$, we get $(5 + 1) \cdot 0.0002 \leqslant 1/10^{n-1}$, or $1.2 \cdot 10^{-3} \leqslant 1/10^{n-1}$. Hence $n = 3$.

602. Determine the number of correct digits in the radius of a circle if its area is known to be equal to 124.35 sq. cm (with an accuracy to within 0.01 sq. cm).

603. Find the limit of the relative error in calculating the total surface of a truncated cone if the radii of its bases are $R = 23.64 \pm 0.01$ (cm), $r = 17.31 \pm \pm 0.01$ (cm), the generator $l = 10.21 \pm 0.01$ (cm); the number $\pi = 3.14$.

604. The number $g = 9.8066$ is an approximate value of the acceleration due to gravity (for the latitude 45°) with five correct digits. Find its relative error.

605. Compute the area of a rectangle whose sides are 92.73 ± 0.01 (m) and 94.5 ± 0.01 (m). Determine the relative error of the result and the number of correct digits.

6.2. The Function of One Independent Variable

Rational and irrational numbers are called real numbers.

The *absolute value* of the real number a is a nonnegative number $|a|$ defined as

follows:

$$|a| = \begin{cases} a & \text{if } a \geqslant 0, \\ -a & \text{if } a < 0. \end{cases}$$

Suppose we are given two nonempty sets X and Y. If there is a certain rule which puts one and only one member y of Y into correspondence with the member x from the set X, then a *function* f (or *mapping*) with a set of values Y is given on the set X. This can be written as $x \in X$, $X \xrightarrow{f} Y$, or f: $X \to Y$, where the set X is called the *domain of definition* of the function, and the set Y, consisting of all numbers of the form $y = f(x)$, the *range* of the function. The domain of definition of the function f is designated as $D(f)$, and the range of the function as $E(f)$. At $x = a$, the value of the function $f(x)$ is denoted by $f(a)$.

In the simplest cases the domain of definition of a function may be an *interval* (*open interval*) (a, b), that is a collection of the values of x complying with the condition $a < x < b$; a *closed interval* $[a, b]$ that is a collection of the values of x satisfying the condition $a \leqslant x \leqslant b$, *semi-open* (*semi-closed*) *interval* $(a, b]$ (i.e. $a < x \leqslant b$) or $[a, b)$ (i.e. $a \leqslant x < b$), an *infinite interval* $(a, +\infty)$ (i.e. $a < x < +\infty$) or $(-\infty, b)$ (i.e. $-\infty < x < b$) or $(-\infty, +\infty)$ (i.e. $-\infty < x < +\infty$); a collection of several open or closed intervals, and the like.

The *graph* of the function $y = f(x)$ is the set of points of the xOy plane with the coordinates $(x, f(x))$, where $x \in X$.

The function $f(x)$ whose domain of definition is symmetrical about zero is called *even* if $f(-x) = f(x)$ for any value of x. The graph of an even function is symmetrical about the axis of ordinates.

The function $f(x)$ whose domain of definition is symmetrical about zero is said to be *odd* if $f(-x) = -f(x)$ for any value of x. The graph of an odd function is symmetrical about the origin.

The function $f(x)$ is called *periodic* if there is a positive number T, called a *period* of the function, such that for all x belonging to the domain of definition of the function the equality $f(x + T) = f(x)$ is satisfied. The *fundamental* (*primitive*) *period* of a function is the least positive number τ possessing the indicated property.

606. Find $\dfrac{f(b) - f(a)}{b - a}$ if $f(x) = x^2$.

Solution. First we find the values of the given function for $x = a$ and $x = b$: $f(a) = a^2$, $f(b) = b^2$ and then we get

$$\frac{f(b) - f(a)}{b - a} = \frac{b^2 - a^2}{b - a} = a + b.$$

607. Find the domain of definition of the function $f(x) = \dfrac{x - 2}{2x - 1}$.

Solution. The given function is defined if $2x - 1 \neq 0$, that is $x \neq 1/2$. Thus, the domain of definition of the function is the union of two intervals: $D(f) = (-\infty, 1/2) \cup (1/2, +\infty)$.

608. Find the domain of definition of the function $f(x) = \dfrac{\ln(1 + x)}{x - 1}$.

Solution. The function is defined if $x - 1 \neq 0$ and $1 + x > 0$, that is if $x \neq 1$ and $x > -1$. The domain of definition of the function is the union of two intervals: $D(f) = (-1, 1) \cup (1, +\infty)$.

609. Find the domain of definition of the function

$$f(x) = \sqrt{1 - 2x} + 3 \arcsin \frac{3x - 1}{2}.$$

Solution. The first summand takes on real values at $1 - 2x \geqslant 0$, and the second, at $-1 \leqslant (3x - 1)/2 \leqslant 1$. Thus, to find the domain of definition of the given function, it is necessary to solve the system of inequalities: $1 - 2x \geqslant 0$, $(3x - 1)/2 \leqslant 1$, $(3x - 1)/2 \geqslant -1$. As a result we get $x \leqslant 1/2$, $x \leqslant 1$, $x \geqslant -1/3$. Consequently, the domain of definition of the function is the closed interval $[-1/3, 1/2]$.

610. Find the range of the functions

(1) $f(x) = x^2 - 6x + 5$; (2) $f(x) = 2 + 3 \sin x$.

Solution. Isolating a perfect square from the quadratic trinomial, we obtain

$$f(x) = x^2 - 6x + 9 - 4 = (x - 3)^2 - 4.$$

For all x's the first summand is a nonnegative number; therefore, the function assumes values not smaller than -4. Thus, the range of the function is the infinite interval $[-4, +\infty)$.

(2) Since the sine assumes values which do not exceed unity in magnitude, we write the inequality $|\sin x| \leqslant 1$, or $-1 \leqslant \sin x \leqslant 1$. Multiplying by 3 all the parts of this double inequality and adding 2 to each of them, we get

$$-3 \leqslant 3 \sin x \leqslant 3; \quad -1 \leqslant 2 + 3 \sin x \leqslant 5.$$

Consequently, $E(f) = [-1, 5]$.

611. Find the fundamental periods of the functions

(1) $f(x) = \cos 8x$; (2) $f(x) = \sin 6x + \tan 4x$.

(1) Since the fundamental period of the function $\cos x$ is 2π, the fundamental period of the function $f(x) = \cos 8x$ is $2\pi/8$, i.e. $\pi/4$.

(2) Here the fundamental period for the first summand is equal to $2\pi/6 = \pi/3$, and for the second summand to $\pi/4$. The fundamental period of the given function is, evidently, the least common multiple of the numbers $\pi/3$ and $\pi/4$, i.e. π.

612. Determine whether the following functions are even or odd:

(1) $f(x) = x^2 \sqrt[3]{x} + 2 \sin x$; (2) $f(x) = 2^x + 2^{-x}$;

(3) $f(x) = |x| - 5e^{x^2}$; (4) $f(x) = x^2 + 5x$;

(5) $f(x) = \log \dfrac{x + 3}{x - 3}$.

Solution. In the examples being considered, the domain of definition of each function is symmetric about zero: in the first four examples $D(f) = (-\infty, +\infty)$ and in the last example $D(f) = (-\infty, -3) \cup (3, +\infty)$.

(1) Substituting $-x$ for x, we get

$$f(-x) = (-x)^2 \sqrt[3]{(-x)} + 2\sin(-x) = -x^2 \sqrt[3]{x} - 2\sin x,$$

i.e. $f(-x) = f(x)$. This means that the function in question is odd.

(2) We have $f(-x) = 2^{-x} + 2^{-(-x)} = 2^{-x} + 2^{x}$, i.e. $f(-x) = f(x)$. Thus, we see that the given function is even.

(3) Here $f(-x) = |x| - 5e^{(-x)^2} = |x| - 5e^{x^2}$, i.e. $f(-x) = f(x)$. Consequently, $f(x)$ is an even function.

(4) We have $f(-x) = (-x)^2 + 5(-x) = x^2 - 5x$. Thus, $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, that is, the function in question is neither even nor odd.

(5) We find

$$f(-x) = \log\frac{-x+3}{-x-3} = \log\frac{x-3}{x+3} = \log\left(\frac{x+3}{x-3}\right)^{-1} = -\log\frac{x+3}{x-3},$$

i.e. $f(-x) = -f(x)$ and, consequently, the function being considered is odd.

613. Find the domains of definition of the following functions:

(1) $f(x) = \sqrt{4 - x^2} + \dfrac{1}{x}$; (2) $f(x) = \arccos\left(\dfrac{x}{2} - 1\right)$;

(3) $f(x) = \dfrac{1}{xe^x}$; (4) $f(x) = \dfrac{x-2}{\cos 2x}$; (5) $f(x) = \dfrac{2x^2 + 3}{x - \sqrt{x^2 - 4}}$;

(6) $f(x) = \log(3x - 1) + 2\log(x + 1)$;

(7) $f(x) = \sqrt{\dfrac{x}{2 - x}} - \sqrt{\sin x}$.

614. Find the ranges of the following functions:

(1) $f(x) = |x| + 1$; (2) $f(x) = 5/x$; (3) $f(x) = \sqrt{16 - x^2}$;

(4) $f(x) = -x^2 + 8x - 13$; (5) $f(x) = 1 - 3\cos x$;

(6) $f(x) = 4^{-x^2}$.

615. Determine whether the following functions are even or odd:

(1) $f(x) = x^4 \sin 7x$; (2) $f(x) = 5|x| - 3\sqrt[3]{x^2}$;

(3) $f(x) = x^4 - 3x^2 + x$; (4) $f(x) = |x| + 2$;

(5) $f(x) = |x + 2|$; (6) $f(x) = \log\cos x$;

(7) $f(x) = \dfrac{16^x - 1}{4^x}$.

616. Find the fundamental periods of the functions

(1) $f(x) = \sin 5x$; (2) $f(x) = -2\cos(x/3) + 1$;

(3) $f(x) = \log\cos 2x$; (4) $f(x) = \tan 3x + \cos 4x$.

6.3. Construction of the Graphs of Functions

The following techniques are used to construct the graphs of functions: using points for construction; operations on the graphs (addition, subtraction, multiplication of the graphs); transformation of the graphs (displacement, stretching).

Proceeding from the graph of the function $y = f(x)$, we can construct the graphs of the functions.

(1) $y = f(x - a)$, which is the original graph displaced by the value a along the x-axis;

(2) $y = f(x) + b$, which is the same graph displaced by the value b along the y-axis;

(3) $y = Af(x)$, which is the original graph stretched A times along the y-axis;

(4) $y = f(kx)$, which is the same graph stretched $1/k$ time along the x-axis.

Thus we can use the graph of the function $y = f(x)$ to construct the graph of a function of the form $y = Af[k(x - a)] + b$.

617. Construct the graph of the function $y = 2x + 1 + \cos x$.

Solution. The graph of the given function can be constructed by adding the graphs of two functions: $y = 2x + 1$ and $y = \cos x$. The graph of the first function is a straight line and it can be constructed using two points, the graph of the second function is a cosine curve (Fig. 23).

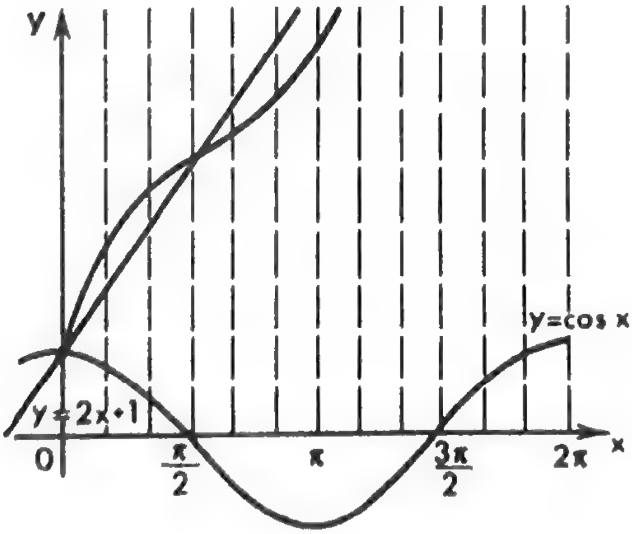

Fig. 23

618. Construct the graph of the function

$$y = \begin{cases} 2 - x & \text{for} \quad x \leqslant 3, \\ 0.1x^2 & \text{for} \quad x > 3. \end{cases}$$

Solution. For $x < 3$ the graph is a ray and for $x \geqslant 3$ it is a branch of a parabola. The sought-for graph is shown in Fig. 24.

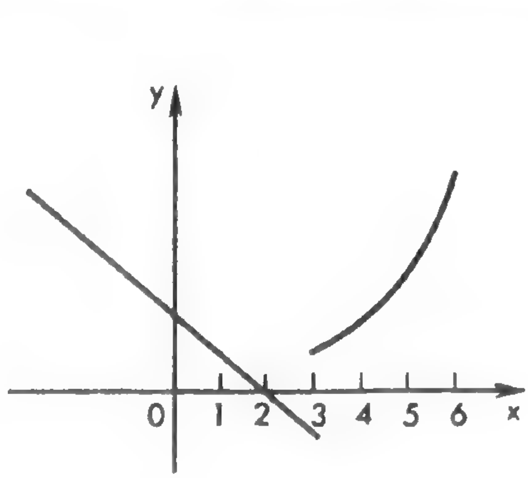

Fig. 24

619. Construct the graph of the function $y = 2\sin(2x - 1)$.

Solution. We reduce the given function to the form $y = 2\sin 2(x - 1/2)$. Here $A = 2$, $k = 2$, $a = 1/2$. Proceeding from the graph $y = \sin x$, we construct the graph of the function $y = \sin 2x$ by twice contracting the original graph along the abscissa axis. Next we construct the graph of the function $y = \sin 2(x - 1/2)$ by displacing the graph obtained by 1/2 to the right. Finally, we stretch the resulting graph twice along the axis of ordinates and get the sought-for graph of the function $y = 2\sin(2x - 1)$ (Fig. 25).

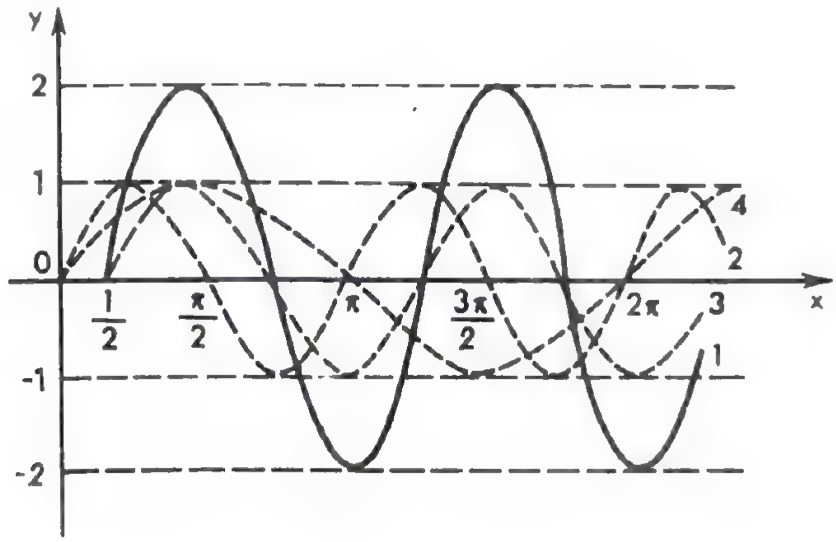

Fig. 25

Construct the graphs of the following functions:

620. $y = \dfrac{x^3 - x}{3}$ on the closed interval $[-4, 4]$.

621. $y = x^2(2 - x)^2$ on the closed interval $[-3, 3]$.

622. $y = \sqrt{x} + \sqrt{4 - x}$ in the domain of definition.

623. $y = 0.5x + 2^{-x}$ on the closed interval $[0, 5]$.
624. $y = 2(x - 1)^3$, proceeding from the function $y = x^3$.

625. $y = \dfrac{1}{x^2 + 4}$. **626.** $y = \dfrac{x^2 + 1}{x}$.

627. $y = \sin(3x - 2) + 1$. **628.** $y = -2\cos(2x + 1)$.
629. $y = \arcsin(x - 2)$.
630. $y = x + 1 + \sin(x - 1)$. **631.** $y \sin x + \cos x$.

632. $y = \begin{cases} -x^2 & \text{for } x < 0, \\ 3x & \text{for } x \geqslant 0. \end{cases}$ **633.** $y = \begin{cases} 4 - x & \text{for } x < -1, \\ 5 & \text{for } -1 \leqslant x \leqslant 0, \\ x^2 + 5 & \text{for } x > 0. \end{cases}$

6.4. Limits

The number a is called the *limit of the sequence*, $x_1, x_2, \ldots, x_n, \ldots$ if for every arbitrarily small positive number ε there is a positive number N such that $|x_n - a| < \varepsilon$ for $n > N$.

In this case we write $\lim\limits_{n \to \infty} x_n = a$.

The number A is said to be the *limit of the function* $f(x)$ as $x \to a$ if for any arbitrarily small $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - A| < \varepsilon$ for $|x - a| < \delta$. The notation is $\lim\limits_{x \to a} f(x) = A$.

Similarly, $\lim\limits_{x \to \infty} f(x) = A$ if $|f(x) - A| < \varepsilon$ for $|x| > N$.

The conventional notation is $\lim\limits_{x \to a} f(x) = \infty$ if $|f(x)| > M$ for $|x - a| < \delta$, where M is an arbitrary positive number.

In this case the function $f(x)$ is said to be an *infinitely large* quantity as $x \to a$.

If $\lim\limits_{x \to a} \alpha(x) = 0$, then the function $\alpha(x)$ is said to be an *infinitesimal* quantity as $x \to a$.

If $x < a$ and $x \to a$, the notation used is $x \to a - 0$; if $x > a$ and $x \to a$, the notation is $x \to a + 0$. The numbers $f(a - 0) = \lim\limits_{x \to a - 0} f(x)$ and $f(a + 0) = \lim\limits_{x \to a + 0} f(x)$ are called, respectively, the *limit on the left* and the *limit on the right* of the function $f(x)$ at a point a.

For the function $f(x)$ to have a limit as $x \to a$ it is necessary and sufficient that $f(a - 0) = f(a + 0)$.

In practice, calculation of limits is based on the following theorems.

If there exist $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$, then

(1) $\lim\limits_{x \to a} [f(x) + g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$;

(2) $\lim\limits_{x \to a} [f(x) \cdot g(x)] = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$;

(3) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$ (for $\lim\limits_{x \to a} g(x) \neq 0$).

The following limits also find use:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \quad (\textit{the first remarkable limit});$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{\alpha\to 0}(1+\alpha)^{1/\alpha} = e = 2.71828\ldots$$ (*the second remarkable limit*).

The logarithm of the number x to the base e is called a *natural logarithm* and is designated as $\ln x$.

When solving the problems, the reader should bear in mind the following equalities:

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1, \quad \lim_{x\to 0}\frac{a^x-1}{x} = \ln a, \quad \lim_{x\to 0}\frac{(1+x)^m-1}{x} = m.$$

634. Show that when $n \to \infty$ the sequence $3, 2\frac{1}{2}, 2\frac{1}{3}, 2\frac{1}{4}, \ldots, 2+\frac{1}{n}, \ldots$ has the number 2 as its limit.

Solution. Here the nth term of the sequence is the number $x_n = 2 + 1/n$. Consequently, $x_n - 2 = 1/n$. Let us preassign a positive number ε. We shall take n large enough for the inequality $1/n < \varepsilon$ to hold true. For that purpose it is sufficient to take $n > 1/\varepsilon$. With this choice of n, we get $|x_n - 2| < \varepsilon$. Hence, $\lim x_n = 2$.

635. Show that when $n \to \infty$ the sequence $7/3, 10/5, 13/7, \ldots, (3n + 4)/(2n + 1), \ldots$ has the number 3/2 as its limit.

Solution. Here $x_n - 3/2 = (3n + 4)/(2n + 1) - 3/2 = 5/[2(2n + 1)]$. Let us find at what value of n the inequality $5/[2(2n + 1)] < \varepsilon$ holds true; since $2(2n + 1) > 5/\varepsilon$ it follows that $n > 5/(4\varepsilon) - 1/2$. Thus we have that if $n > 5/(4\varepsilon) - 1/2$, then $|x_n - 3/2| < \varepsilon$, i.e. $\lim\limits_{n\to\infty} x = 3/2$.

Putting $\varepsilon = 0.1$, we infer that the inequality $|x_n - 3/2| < 0.1$ is satisfied at $n > 12$ (for instance at $n = 13$). Similarly, the inequality $|x_n - 3/2| < 0.01$ is satisfied at $n > 124.5$ (for instance at $n = 125$), and the inequality $|x_n - 3/2| < 0.001$ at $n > 1249.5$ (for instance at $n = 1250$).

636. Find $\lim\limits_{x\to 4}\dfrac{5x+2}{2x+3}$.

Solution. Since $x \to 4$, the numerator of the fraction tends to the number $5\cdot 4 + 2 = 22$, and the denominator, to the number $2\cdot 4 + 3 = 11$. Consequently, $\lim\limits_{x\to 4}\dfrac{5x+2}{2x+3} = \dfrac{22}{11} = 2.$

637. Find $\lim\limits_{x\to\infty}\dfrac{3x+5}{2x+7}$.

Solution. The numerator and the denominator of the fraction increase indefinitely as $x \to \infty$. In such a case, it is said that an indeterminacy of the form ∞/∞ obtains. Dividing the numerator and the denominator of the fraction by x, we get

$$\lim_{x\to\infty}\frac{3x+5}{2x+7}=\lim_{x\to\infty}\frac{3+5/x}{2+7/x}=\frac{3}{2},$$

since each of the fractions $5/x$ and $7/x$ tends to zero as $x\to\infty$.

638. Find $\lim\limits_{x\to 3}\dfrac{x^2-9}{x^2-3x}$.

Solution. Here, as $x\to 3$, the numerator and the denominator of the fraction tend to zero (it is customary to say that an indeterminacy of the form 0/0 obtains). It follows that

$$\frac{x^2-9}{x^2-3x}=\frac{(x-3)(x+3)}{x(x-3)}=\frac{x+3}{x};$$

if $x\neq 3$, then $\lim\limits_{x\to 3}\dfrac{x^2-9}{x^2-3x}=\lim\limits_{x\to 3}\dfrac{x+3}{x}$. But when $x\to 3$ the fraction $\dfrac{x+3}{x}$ tends to the number $\dfrac{3+3}{3}=2$. Thus, $\lim\limits_{x\to 3}\dfrac{x^2-9}{x^2-3x}=2$.

639. Find $\lim\limits_{x\to 1}\dfrac{x^3-x^2-x+1}{x^3+x^2-x-1}$.

Solution. Here we have an indeterminacy of the form 0/0. Let us factorize the numerator and the denominator of the fraction:

$$\lim_{x\to 1}\frac{x^3-x^2-x+1}{x^3+x^2-x-1}=\lim_{x\to 1}\frac{x^2(x-1)-(x-1)}{x^2(x+1)-(x+1)}$$
$$=\lim_{x\to 1}\frac{(x-1)^2(x+1)}{(x-1)(x+1)^2}=\lim_{x\to 1}\frac{x-1}{x+1}=\frac{0}{2}=0.$$

640. Find $\lim\limits_{x\to 10}\dfrac{x^3-1000}{x^3-20x^2+100x}$.

Solution. This is also an indeterminacy of the form 0/0. We have

$$\lim_{x\to 10}\frac{x^3-1000}{x^3-20x^2+100x}=\lim_{x\to 10}\frac{(x-10)(x^2+10x+100)}{x(x-10)^2}$$
$$=\lim_{x\to 10}\frac{x^2+10x+100}{x(x-10)}.$$

The numerator of the fraction tends to 300 and the denominator to zero, that is, the denominator is an infinitesimal quantity. Consequently, the fraction in question is an infinitely large quantity and $\lim\limits_{x\to 10}\dfrac{x^3-1000}{x^3-2x^2+100x}=\infty$.

641. Find $\lim\limits_{x\to 0}\dfrac{\sqrt{x+4}-2}{x}$.

Solution. Let us multiply the numerator and the denominator of the fraction by the sum $\sqrt{x+4}+2$:

$$\lim_{x\to 0}\frac{(\sqrt{x+4}-2)(\sqrt{x+4}+2)}{x(\sqrt{x+4}+2)}=\lim_{x\to 0}\frac{x+4-4}{x(\sqrt{x+4}+2)}=\lim_{x\to 0}\frac{1}{\sqrt{x+4}+2}=\frac{1}{4}.$$

642. Find $\lim\limits_{x\to 0}\dfrac{\sqrt[5]{(1+x)^3}-1}{x}$.

Solution. Let us put $1+x=y^5$. Then we have

$$\lim_{x\to 0}\frac{\sqrt[5]{(1+x)^3}-1}{x}=\lim_{y\to 1}\frac{y^3-1}{y^5-1}=\lim_{y\to 1}\frac{y^2+y+1}{y^4+y^3+y^2+y+1}=\frac{3}{5}.$$

643. Find $\lim\limits_{x\to 0}\dfrac{\sin mx}{x}$.

Solution. Using the first remarkable limit, we obtain

$$\lim_{x\to 0}\frac{\sin mx}{x}=\lim_{x\to 0}\frac{m\cdot\sin mx}{mx}=m\cdot\lim_{x\to 0}\frac{\sin mx}{mx}=m.$$

644. Find $\lim\limits_{x\to 0}\dfrac{1-\cos 5x}{x^2}$.

Solution. We have

$$\lim_{x\to 0}\frac{1-\cos 5x}{x^2}=\lim_{x\to 0}\frac{2\sin^2(5x/2)}{x^2}=2\lim_{x\to 0}\left(\frac{\sin(5x/2)}{x}\right)^2=2\cdot\left(\frac{5}{2}\right)^2=\frac{25}{2}.$$

We have used here the results of the preceding example, taking $m=5/2$.

645. Find $\lim\limits_{x\to\infty}\dfrac{x^3+2x^2+3x+4}{4x^3+3x^2+2x+1}$.

Solution. This is an indeterminacy of the form ∞/∞. Let us divide the numerator and the denominator of the fraction by the leading power of x, i.e. by x^3:

$$\lim_{x\to\infty}\frac{x^3+2x^2+3x+4}{4x^3+3x^2+2x+1}=\lim_{x\to\infty}\frac{1+2/x+3/x^2+4/x^3}{4+3/x+2/x^2+1/x^3}=\frac{1}{4}.$$

646. Find $\lim\limits_{x\to\infty}\dfrac{3x^4-2}{\sqrt{x^8+3x+4}}$.

Solution. Let us divide the numerator and the denominator by x^4:

$$\lim_{x\to\infty}\frac{3x^4-2}{\sqrt{x^8+3x+4}}=\lim_{x\to\infty}\frac{3-2/x^4}{\sqrt{1+3/x^7+4/x^8}}=\frac{3}{1}=3.$$

647. Find $\lim\limits_{x\to\infty}(\sqrt{x^2+8x+3}-\sqrt{x^2+4x+3})$.

Solution. Here we have an indeterminacy of the form $\infty-\infty$.

Multiply and divide the given expression by $\sqrt{x^2 + 8x + 3} + \sqrt{x^2 + 4x + 3}$:

$$\lim_{x \to \infty} (\sqrt{x^2 + 8x + 3} - \sqrt{x^2 + 4x + 3})$$

$$= \lim_{x \to \infty} \frac{(\sqrt{x^2 + 8x + 3} - \sqrt{x^2 + 4x + 3})(\sqrt{x^2 + 8x + 3} + \sqrt{x^2 + 4x + 3})}{\sqrt{x^2 + 8x + 3} + \sqrt{x^2 + 4x + 3}}$$

$$= \lim_{x \to \infty} \frac{x^2 + 8x + 3 - x^2 - 4x - 3}{\sqrt{x^2 + 8x + 3} + \sqrt{x^2 + 4x + 3}} = \lim_{x \to \infty} \frac{4x}{\sqrt{x^2 + 8x + 3} + \sqrt{x^2 + 4x + 3}}$$

$$= \lim_{x \to \infty} \frac{4}{\sqrt{1 + 8/x + 3/x^2} + \sqrt{1 + 4/x + 3/x^2}} = \frac{4}{2} = 2.$$

648. Find $\lim\limits_{x \to \infty} \left(\dfrac{x^2 + 5x + 4}{x^2 - 3x + 7} \right)^x$.

Solution. By means of division of the numerator of the fraction by the denominator, we isolate the integral part:

$$\frac{x^2 + 5x + 4}{x^2 - 3x + 7} = 1 + \frac{8x - 3}{x^2 - 3x + 7}.$$

Thus, when $x \to \infty$ the given function is a power whose base tends to unity and the exponent, to infinity (an indeterminacy of the form 1^∞). Let us transform the function so that we can use the second remarkable limit. We obtain

$$\lim_{x \to \infty} \left(\frac{x^2 + 5x + 4}{x^2 - 3x + 7} \right)^x = \lim_{x \to \infty} \left(1 + \frac{8x - 3}{x^2 - 3x + 7} \right)^x$$

$$= \lim_{x \to \infty} \left[\left(1 + \frac{8x - 3}{x^2 - 3x + 7} \right)^{\frac{x^2 - 3x + 7}{8x - 3}} \right]^{\frac{x(8x - 3)}{x^2 - 3x + 7}}$$

$$= \lim_{x \to \infty} \left[\left(1 + \frac{8x - 3}{x^2 - 3x + 7} \right)^{\frac{x^2 - 3x + 7}{8x - 3}} \right]^{\frac{8 - 3/x}{1 - 3/x + 7/x^2}}.$$

Since $\dfrac{8x - 3}{x^2 - 3x + 7} \to 0$ as $x \to \infty$, we have

$$\lim_{x \to \infty} \left(1 + \frac{8x - 3}{x^2 - 3x + 7} \right)^{\frac{x^2 - 3x + 7}{8x - 3}} = e.$$

Taking into account that $\lim\limits_{x \to \infty} \dfrac{8 - 3/x}{1 - 3/x + 7/x^2} = 8$, we get

$$\lim_{x \to \infty} \left(\frac{x^2 + 5x + 4}{x^2 - 3x + 7} \right)^x = e^8.$$

649. Find the limits on the left and on the right of the function $f(x) = \dfrac{1}{x + 2^{1/(x-3)}}$ as $x \to 3$.

Solution. If $x \to 3 - 0$, then $1/(x - 3) \to -\infty$ and $2^{1/(x-3)} \to 0$. Consequently, $\lim\limits_{x \to 3 - 0} f(x) = 1/3$. Now if $x \to 3 + 0$, then $1/(x - 3) \to +\infty$, $2^{1/(x-3)} \to +\infty$ and $\lim\limits_{x \to 3 + 0} f(x) = 0$.

650. Find the limits on the left and on the right of the function $f(x) = e^{1/(x-a)}$ as $x \to a$.

Solution. If $x \to a - 0$, then $1/(x - a) \to -\infty$ and $\lim\limits_{x \to a - 0} f(x) = 0$. Now if $x \to a + 0$, then $1/(x - a) \to +\infty$ and $\lim\limits_{x \to a + 0} f(x) = +\infty$.

651. Show that when $n \to \infty$ the sequence $1/2, 5/3, 9/4, \ldots, (4n - 3)/(n + 1), \ldots$, has a limit equal to 4.

652. Show that when $n \to \infty$ the sequence $1, 1/3, 1/5, \ldots, 1/(2n - 1), \ldots$ is an infinitesimal quantity.

Find the following limits:

653. $\lim\limits_{x \to 2} \dfrac{x^2 - 6x + 8}{x^2 - 8x + 12}$. 654. $\lim\limits_{x \to 0} \dfrac{\sqrt{1 + x + x^2} - \sqrt{1 - x + x^2}}{x^2 - x}$.

655. $\lim\limits_{x \to 3} \dfrac{x^2 - 5x + 6}{x^2 - 9}$. 656. $\lim\limits_{x \to 1} \dfrac{x^3 - 6x^2 + 11x - 6}{x^2 - 3x + 2}$.

657. $\lim\limits_{h \to 0} \dfrac{\sin(a + 2h) - 2\sin(a + h) + \sin a}{h^2}$. 658. $\lim\limits_{x \to 0} \dfrac{\tan mx}{\sin nx}$.

659. $\lim\limits_{x \to x_0} \dfrac{\tan x - \tan x_0}{x - x_0}$.

660. $\lim\limits_{x \to \pi/4} \dfrac{\sin x - \cos x}{\pi - 4x}$. 661. $\lim\limits_{x \to \pi/2} \dfrac{\cos x}{\pi - 2x}$.

662. $\lim\limits_{x \to \infty} \dfrac{2x^4 + 3x^2 + 5x - 6}{x^3 + 3x^2 + 7x - 1}$.

Hint. Put $\pi/2 - x = \alpha$.

663. $\lim\limits_{x \to \infty} \dfrac{(2x^3 + 4x + 5)(x^2 + x + 1)}{(x + 2)(x^4 + 2x^3 + 7x^2 + x - 1)}$. 664. $\lim\limits_{x \to 2} \dfrac{x^2 - 7x + 10}{x^2 - 8x + 12}$.

665. $\lim\limits_{x \to -1} \dfrac{\sqrt{4 + x + x^2} - 2}{x + 1}$. 666. $\lim\limits_{x \to 0} \dfrac{\sqrt{1 + x\sin x} - 1}{x^2}$.

667. $\lim\limits_{x \to 0} \dfrac{x}{\sqrt[3]{1 + x} - 1}$. 668. $\lim\limits_{x \to 2} \dfrac{\sqrt{1 + x + x^2} - \sqrt{7 + 2x - x^2}}{x^2 - 2x}$.

669. $\lim\limits_{x \to 0} \dfrac{1 - \cos 5x}{1 - \cos 3x}$. 670. $\lim\limits_{x \to 0} \dfrac{\tan x - \sin x}{x^3}$.

671. $\lim\limits_{x\to\infty}\dfrac{\ln(1+mx)}{x}$. **672.** $\lim\limits_{x\to\pm\infty}\dfrac{2^x+3}{2^x-3}$.

673. $\lim\limits_{x\to\infty}(\sqrt{x^2+ax+b}-\sqrt{x^2+cx+d})$.

674. $\lim\limits_{x\to\infty}(\sin\sqrt{x+1}-\sin\sqrt{x})$.

675. $\lim\limits_{x\to\infty}(\sqrt[3]{x+1}-\sqrt[3]{x})$. **676.** $\lim\limits_{x\to0}\dfrac{1-5^x}{1-e^x}$.

677. $\lim\limits_{x\to0}\dfrac{8^x-7^x}{6^x-5^x}$. **678.** $\lim\limits_{x\to0}\dfrac{\sin 2x}{\ln(1+x)}$.

679. $\lim\limits_{x\to0}\dfrac{5^x-1}{x}$. **680.** $\lim\limits_{x\to1}\dfrac{\sqrt[4]{x}-1}{x-1}$.

Hint. Put $x = t^4$.

681. $\lim\limits_{x\to\pm0}\dfrac{\sin x}{|x|}$. **682.** $\lim\limits_{t\to0}\dfrac{t+\sin t}{t-\sin t}$. **683.** $\lim\limits_{x\to0}\dfrac{\sin 3x-\sin x}{\ln(x+1)}$.

684. $\lim\limits_{x\to5-0}10^{1/(x-5)}$. **685.** $\lim\limits_{x\to\infty}\sin x$. **686.** $\lim\limits_{x\to1}\dfrac{x^5-1}{x^4-1}$.

687. Find $\lim\limits_{t\to\infty}t(\sqrt[t]{a}-1)$ (where $t>0$).

Hint. Put $x = 1/t$, where $x\to0$.

688. $\lim\limits_{x\to\infty}\left(\dfrac{x^2+1}{x^2}\right)^{x^2+1}$. **689.** $\lim\limits_{x\to\infty}\left(1+\dfrac{3}{x}\right)^x$.

690. $\lim\limits_{x\to3}\left(\dfrac{1}{x-3}-\dfrac{6}{x^2-9}\right)$. **691.** $\lim\limits_{x\to0}\dfrac{5^x-4^x}{x^2+x}$.

Hint. Reduce the fractions to the common denominator.

692. $\lim\limits_{x\to1}\dfrac{x^x-1}{x\ln x}$. **693.** $\lim\limits_{x\to0}\dfrac{\ln(1-3x)}{x}$.

Hint. Take into account that $x^x = e^{x\ln x}$.

694. $\lim\limits_{x\to\infty}\dfrac{x^4+5x^3+7}{2x^5+3x^4+1}$.

695. $\lim\limits_{x\to0}\dfrac{\ln(x+2)-\ln 2}{x}$. **696.** $\lim\limits_{x\to\infty}\left(\dfrac{x+8}{x-2}\right)^x$.

697. $\lim\limits_{\alpha\to0}(2-\cos\alpha)^{\operatorname{cosec}^2\alpha}$. **698.** Find $\lim\limits_{x\to\infty}\left(\dfrac{x+a}{x+b}\right)^{x+c}$.

699. $\lim\limits_{x\to2}\left(\dfrac{x}{2}\right)^{1/(x-2)}$.

6.5. Comparison of Infinitesimal Quantities

Suppose $\alpha(x)$ and $\beta(x)$ are infinitesimal quantities as $x \to a$.

1. If $\lim\limits_{x \to a} \dfrac{\alpha}{\beta} = 0$, then α is said to be an *infinitesimal quantity of the higher order of smallness* as compared to β. In this case we write $\alpha = o(\beta)$.

2. If $\lim\limits_{x \to a} \dfrac{\alpha}{\beta} = m$, where m is a nonzero number, then α and β are said to be *infinitesimal quantities of the same order of smallness*. In particular, if $\lim\limits_{x \to a} \dfrac{\alpha}{\beta} = 1$, then the infinitesimal quantities α and β are said to be *equivalent*. The notation $\alpha \sim \beta$ signifies that α and β are equivalent infinitesimal quantities.

If $\dfrac{\alpha}{\beta} \to \infty$, this means that $\lim \dfrac{\beta}{\alpha} = 0$. Thus, β is an infinitesimal of the higher order of smallness compared to α, i.e. $\beta = o(\alpha)$.

3. If α^k and β are infinitesimals of the same order of smallness, with $k > 0$, then the infinitesimal β is said to have the *order* k as compared to α.

Here are some properties of infinitesimal quantities:

1°. *The product of two infinitesimals is an infinitesimal of the higher order of smallness compared to the multipliers*, that is, if $\gamma = \alpha\beta$, then $\gamma = o(\alpha)$ and $\gamma = o(\beta)$.

2° *The infinitesimals α and β are equivalent if and only if their difference $\alpha - \beta = \gamma$ is an infinitesimal of the higher order of smallness as compared to α and β*, that is, if $\gamma = o(\alpha)$, $\gamma = o(\beta)$.

3°. *If the ratio of two infinitesimals has a limit, then this limit will not change upon a replacement of each infinitesimal by an equivalent infinitesimal*, that is, if $\lim\limits_{x \to a} \dfrac{\alpha}{\beta} = m$, $\alpha \sim \alpha_1$, $\beta \sim \beta_1$, then $\lim\limits_{x \to a} \dfrac{\alpha_1}{\beta_1} = m$.

It may be useful to bear in mind the equivalence of the following infinitesimal quantities: if $x \to 0$, then $\sin x \sim x$, $\tan x \sim x$, $\arcsin x \sim x$, $\arctan x \sim x$, $\ln(1 + x) \sim x$.

700. Assume t to be an infinitesimal quantity. Compare the infinitesimal $\alpha = 5t^2 + 2t^5$ and $\beta = 3t^2 + 2t^3$.

Solution. We find

$$\lim_{t \to 0} \frac{\alpha}{\beta} = \lim_{t \to 0} \frac{5t^2 + 2t^5}{3t^2 + 2t^3} = \lim_{t \to 0} \frac{5 + 2t^3}{3 + 2t} = \frac{5}{3}.$$

Since the limit of the ratio of α to β is a number different from zero, then α and β are infinitesimals of the same order of smallness.

701. Compare the infinitesimal quantities $\alpha = t\sin^2 t$ and $\beta = 2t\sin t$ as $t \to 0$.

Solution. We find

$$\lim_{t \to 0} \frac{\alpha}{\beta} = \lim_{t \to 0} \frac{t\sin^2 t}{2t\sin t} = \frac{1}{2} \lim_{t \to 0} \sin t = 0,$$

i.e. $\alpha = o(\beta)$.

702. Compare the infinitesimal quantities $\alpha = t\ln(1 + t)$ and $\beta = t\sin t$ as $t \to 0$.

Solution. We have

$$\lim_{t \to 0} \frac{\alpha}{\beta} = \lim_{t \to 0} \frac{t\ln(1 + t)}{t\sin t} = \lim_{t \to 0} \frac{\ln(1 + t)}{\sin t} = \lim_{t \to 0} \frac{\dfrac{\ln(1 + t)}{t}}{\dfrac{\sin t}{t}} = 1,$$

i.e. $\alpha \sim \beta$.

703. Find $\lim\limits_{x \to 0} \dfrac{\ln(1 + 3x\sin x)}{\tan x^2}$.

Solution. Replacing the numerator and the denominator of the fraction by equivalent infinitesimals $\ln(1 + 3x\sin x) \sim 3x\sin x$, $\tan x^2 \sim x^2$, we obtain

$$\lim_{x \to 0} \frac{\ln(1 + 3x\sin x)}{\tan x^2} = \lim_{x \to 0} \frac{3x\sin x}{x^2} = 3 \lim_{x \to 0} \frac{\sin x}{x} = 3.$$

704. Determine the order of smallness of the infinitesimal $y = xe^x$ in comparison with the infinitesimal quantity x.

705. Determine the order of smallness of the infinitesimal $y = \sqrt{1 + x\sin x} - 1$ in comparison with the infinitesimal quantity x.

706. Determine the order of smallness of the infinitesimal $y = \sqrt{\sin 2x}$ as compared to x.

707. Compare the infinitesimal quantities $\alpha = t^2\sin^2 t$ and $\beta = t \tan t$ if $t \to 0$.

708. Compare the infinitesimals $\alpha = (1 + x)^m - 1$ and $\beta = mx$ if $x \to 0$ and m is a rational positive number.

709. Compare the infinitesimal quantities $\alpha = a^x - 1$ and $\beta = x \ln a$, if $x \to 0$.

Find the following limits.

710. $\lim\limits_{x \to 0} \dfrac{\sqrt{1 + 2x} - 1}{\tan 3x}$. **711.** $\lim\limits_{x \to 0} \dfrac{\sin^2 3x}{\ln^2(1 + 2x)}$.

Hint. Replace the numerator and the denominator by equivalent infinitesimals.

712. $\lim\limits_{x \to 0} \dfrac{e^{2x} - 1}{\ln(1 - 4x)}$.

713. $\lim\limits_{x \to 1} \dfrac{\ln(1 + x - 3x^2 + 2x^3)}{\ln(1 + 3x - 4x^2 + x^3)}$. **714.** $\lim\limits_{x \to 0} \dfrac{\ln \cos x}{\ln(1 + x^2)}$.

Hint. Represent $\cos x$ in the form $1 - (1 - \cos x)$.

715. $\lim\limits_{x \to 1} \dfrac{\sin (e^{x - 1} - 1)}{\ln x}$.

716. $\lim\limits_{x \to 0} \dfrac{\sqrt[3]{(1 + x)^3} - 1}{(1 + x)\sqrt[3]{(1 + x)^2} - 1}$.

717. $\lim\limits_{\alpha \to 0} \dfrac{(5^\alpha - 1)(4^\alpha - 1)}{(3^\alpha - 1)(6^\alpha - 1)}$.

718. Find $\lim\limits_{x \to 0} \dfrac{\sqrt[3]{8 + 3x} - 2}{\sqrt[4]{16 + 5x} - 2}$.

Hint. Divide the numerator and the denominator by 2.

6.6. Continuity of a Function

A function $f(x)$ is said to be *continuous at a point a* if: (1) this function is defined in a certain neighbourhood of the point a; (2) there is a limit $\lim\limits_{x \to a} f(x)$; (3) this limit is equal to the value of the function at the point a, i.e. $\lim\limits_{x \to a} f(x) = f(a)$.

Using the notation $x - a = \Delta x$ (increment of the argument) and $f(x) - f(a) = \Delta y$ (increment of the function), the condition of continuity can be written as follows: $\lim\limits_{\Delta x \to 0} \Delta y = 0$, that is, *the function $f(x)$ is continuous at the point a if and only if at that point an infinitesimal increment of the function corresponds to an infinitesimal increment of the argument.*

If a function is continuous at every point of a certain domain (interval, closed interval, etc.), it is said to be *continuous in that domain.*

The point a belonging to the domain of definition of the function or being a boundary point for that domain is called a *point of discontinuity* if the condition of continuity of the function is violated at that point.

If there exist both the left-hand and right-hand limits $\lim\limits_{x \to a - 0} f(x) = f(a - 0)$ and $\lim\limits_{x \to a + 0} f(x) = f(a + 0)$, not all the three numbers $f(a)$, $f(a - 0)$, $f(a + 0)$ being equal, then a is called a *point of discontinuity of the 1st kind.*

Points of discontinuity of the first kind are divided, in their turn, into *points of removable discontinuity* (when $f(a - 0) = f(a + 0) \neq f(a)$, i.e. the left-hand limit and the right-hand limit of the function at the point a are equal to each other but are not equal to the value of the function at that point), *points of jump discontinuity* (when $f(a - 0) \neq f(a + 0)$, i.e. the left-hand limit of the function at the point a differs from the right-hand limit); in the latter case the difference $f(a + 0) - f(a - 0)$ is called a *jump discontinuity* of the function at the point a.

Points of discontinuity which are not the points of discontinuity of the 1st kind are called *points of discontinuity of the 2nd kind.* At these points either the left-hand limit or the right-hand limit does not exist.

The sum and the product of a finite number of continuous functions are continuous functions. The quotient of two continuous functions is a continuous function at all the points at which the divisor is nonzero.

719. Show that for $x = 4$ the function $y = \dfrac{x}{x - 4}$ possesses a discontinuity.

Sotution. We find

$$\lim_{x \to 4 - 0} \frac{x}{x - 4} = -\infty, \quad \lim_{x \to 4 + 0} \frac{x}{x - 4} = +\infty.$$

Thus, as $x \to 4$ the function possesses neither the left-hand nor the right-hand limit. Consequently, $x = 4$ is a point of discontinuity of the 2nd kind (Fig. 26).

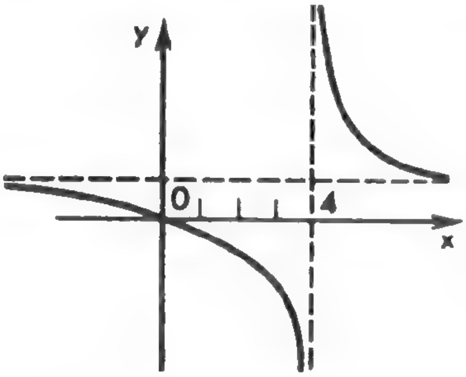

Fig. 26

720. Show that for $x = 4$ the function $y = \arctan \dfrac{1}{x - 4}$ possesses a discontinuity.

Solution. If $x \to 4 - 0$, then $1/(x - 4) \to -\infty$ and $\lim\limits_{x \to 4 - 0} y = -\pi/2$. Now if $x \to 4 + 0$, then $1/(x - 4) \to +\infty$ and $\lim\limits_{x \to 4 + 0} y = \pi/2$. Thus, as $x \to 4$, the function possesses both the left-hand and the right-hand limit and these limits are different. Consequently, $x = 4$ is a point of discontinuity of the 1st kind, a point of jump discontinuity. The jump discontinuity of the function at that point is equal to $\pi/2 - (-\pi/2) = \pi$ (Fig. 27).

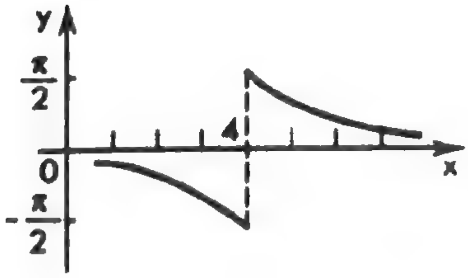

Fig. 27

721. Show that for $x = 5$ the function $y = \dfrac{x^2 - 25}{x - 5}$ has a discontinuity.

Solution. The function is not defined at the point $x = 5$ since upon a substitution we get an indeterminacy 0/0. At other points the fraction can be cancelled by $x - 5$ because $x - 5 \neq 0$. Consequently, $y = x + 5$ at $x \neq 5$. It is easy to see that $\lim\limits_{x \to 5 - 0} y = \lim\limits_{x \to 5 + 0} y = 10$.

Thus, at $x = 5$ the function possesses a removable discontinuity. It will be removed if we assume that $y = 10$ for $x = 5$.

Thus, we can consider the function $y = (x^2 - 25)/(x - 5)$ to be continuous at all the values of x if we assume that the equality $(x^2 - 25)/(x - 5) = x + 5$ is valid for all the values of x, the value $x = 5$ inclusive. In this case the graph of the function is the straight line $y = x + 5$.

722. Find the points of discontinuity of the function $y = \dfrac{2^{1/(x-2)} - 1}{2^{1/(x-2)} + 1}$.

723. Find the points of discontinuity of the function $y = \dfrac{1}{(x-1)(x-5)}$.

724. What is the discontinuity of the function $y = \dfrac{1}{1 - e^{1-x}}$ at the point $x = 1$?

725. What is the discontinuity of the function $y = \dfrac{\sin x}{x}$ at the point $x = 0$?

726. Find the points of discontinuity of the function $y = \dfrac{\tan x \operatorname{arc\,tan} \dfrac{1}{x-3}}{x(x-5)}$.

727. Find the points of discontinuity of the function

$$y = \frac{x^3 - 6x^2 + 11x - 6}{x^2 - 3x + 2}.$$

728. Find the points of discontinuity of the function $y = \dfrac{x+1}{x^3 + 6x^2 + 11x + 6}$.

729. Find the points of discontinuity of the function $y = \dfrac{1}{x^2 + x + 1}$.

730. Test the function $y = \dfrac{1}{(x-1)(x-6)}$ for continuity on the closed interval (1) $[2, 5]$; (2) $[4, 10]$; (3) $[0, 7]$.

731. Test the function $y = \dfrac{1}{x^4 - 26x^2 + 25}$ for continuity on the closed interval (1) $[6, 10]$; (2) $[-2, 2]$; (3) $[-6, 6]$.

Chapter 7

Differential Calculus of Functions of One Independent Variable

7.1. The Derivative and the Differential

7.1.1. Differentiation of explicit functions. Suppose x_1 and x_2 are the values of the argument and $y_1 = f(x_1)$ and $y_2 = f(x_2)$ are the corresponding values of the function $y = f(x)$. The difference $\Delta x = x_2 - x_1$ is called the *increment of the argument* and the difference $\Delta y = y_2 - y_1 = f(x_2) - f(x_1)$, the *increment of the function* on the closed interval $[x_1, x_2]$.

The *derivative* of the function $y = f(x)$ with respect to the argument x is the limit of the ratio between the increment of the function and the increment of the argument when the latter tends to zero:

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}, \text{ or } f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

(the derivative can be also designated as dy/dx).

In geometrical interpretation the derivative is the slope of the tangent to the graph of the function $y = f(x)$ at the point x, i.e. $y' = \tan \alpha$.

The derivative is the *rate of change of the function* at the point x.

A search for the derivative is the *differentiation* of the function.

Formulas for differentiation of the basic functions

I. $(x^m)' = mx^{m-1}$.

II. $(\sqrt{x})' = \dfrac{1}{2\sqrt{x}}$.

III. $\left(\dfrac{1}{x}\right)' = -\dfrac{1}{x^2}$.

IV. $(e^x)' = e^x$.

V. $(a^x)' = a^x \ln a$.

VI. $(\ln x)' = \dfrac{1}{x}$.

VII. $(\log_a x)' = \dfrac{1}{x \ln a}$.

VIII. $(\sin x)' = \cos x$.

IX. $(\cos x)' = -\sin x$.

X. $(\tan x)' = \sec^2 x$.

XI. $(\cot x)' = -\operatorname{cosec}^2 x$.

XII. $(\arcsin x)' = \dfrac{1}{\sqrt{1 - x^2}}$.

XIII. $(\arccos x)' = -\dfrac{1}{\sqrt{1 - x^2}}$.

XIV. $(\arctan x)' = \dfrac{1}{1 + x^2}$.

XV. $(\arctan x)' = -\dfrac{1}{1 + x^2}$.

XVI. $(\sinh x)' = \left(\dfrac{e^x - e^{-x}}{2}\right)' = \cosh x$.

XVII. $(\cosh x)' = \left(\frac{e^x + e^{-x}}{2}\right)' = \sinh x.$

XVIII. $(\tanh x)' = \left(\frac{\sinh x}{\cosh x}\right) = \frac{1}{\cosh^2 x}.$

XIX. $(\coth x)' = \left(\frac{\cosh x}{\sinh x}\right) = -\frac{1}{\sinh^2 x}.$

Principal rules of differentiation.

Suppose C is a constant and $u = u(x)$, $v = v(x)$ are functions possessing derivatives. Then:

$$(1)\ C' = 0;\ (2)\ x' = 1;\ (3)\ (u \pm v)' = u' \pm v';\ (4)\ (Cu)' = Cu';$$

$$(5)\ (uv)' = u'v + uv';\ (6)\ \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2};$$

(7) if $y = f(u)$, $u = u(x)$, i.e. $y = f[u(x)]$, where the functions $f(u)$ and $u(x)$ possess derivatives, then

$$y'_x = y'_u \cdot u'_x$$

(rule of differentiation of a ***composite function***).

732. Proceeding from the definition of the derivative (without using differentiation formulas), find the derivative of the function $y = 2x^3 + 5x^2 - 7x - 4$.

Solution. We assign to x an increment Δx. Then y will receive the increment Δy:

$$y + \Delta y = 2(x + \Delta x)^3 + 5(x + \Delta x)^2 - 7(x + \Delta x) - 4.$$

We get the increment of the function:

$$\Delta y = [2(x + \Delta x)^3 + 5(x + \Delta x)^2 - 7(x + \Delta x) - 4] - (2x^3 + 5x^2 - 7x - 4) =$$
$$= 6x^2\Delta x + 6x\Delta x^2 + 2\Delta x^3 + 10x\Delta x + 5\Delta x^2 - 7\Delta x.$$

Next we find the ratio between the increment of the function and the increment of the argument:

$$\frac{\Delta y}{\Delta x} = 6x^2 + 6x\Delta x + 2\Delta x^2 + 10x + 5\Delta x - 7.$$

Then we find the limit of this ratio as $\Delta x \to 0$:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (6x^2 + 6x\Delta x + 2\Delta x^2 + 10x + 5\Delta x - 7) =$$
$$= 6x^2 + 10x - 7.$$

Consequently, by the definition of the derivative $y' = 6x^2 + 10x - 7$.

733. Proceeding from the definition of the derivative find the derivative of the function $y = \sqrt{x}$.

Solution. We find the increment of the function: $\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$. From this it follows that

$$\frac{\Delta y}{\Delta x} = \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}.$$

Consequently

$$y' = \lim_{\Delta x \to 0} \frac{(\sqrt{x + \Delta x} - \sqrt{x})(\sqrt{x + \Delta x} + \sqrt{x})}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} = \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} =$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Thus we have $y' = \dfrac{1}{2\sqrt{x}}$.

734. Proceeding from the definition of the derivative, find the derivative of the function $y = -\cot x - x$.

Solution. We find

$$\Delta y = -\cot (x + \Delta x) - (x + \Delta x) + \cot x + x = \cot x - \cot (x + \Delta x) - \Delta x.$$

Using the formula $\cot \alpha - \cot \beta = \dfrac{\sin(\beta - \alpha)}{\sin \alpha \sin \beta}$, we get

$$\Delta y = \frac{\sin(x + \Delta x - x)}{\sin x \sin(x + \Delta x)} - \Delta x = \frac{\sin \Delta x}{\sin x \sin(x + \Delta x)} - \Delta x,$$

whence

$$\frac{\Delta y}{\Delta x} = \frac{\frac{\sin \Delta x}{\Delta x}}{\sin x \sin(x + \Delta x)} - 1$$

and, consequently,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{\sin \Delta x}{\Delta x}}{\sin x \cdot \sin (x + \Delta x)} - 1 = \frac{1}{\sin^2 x} - 1.$$

Thus we have $y' = \dfrac{1}{\sin^2 x} - 1 = \cot^2 x.$

Proceeding from the definition of the derivative, find the derivatives of the functions:

735. $y = \dfrac{1}{x^2}$. **736.** $y = \sqrt[3]{x^2}$. **737.** $y = 5 \sin x + 3 \cos x$.

738. $y = 5(\tan x - x)$. **739.** $y = \dfrac{1}{e^x + 1}$. **740.** $y = 2^{x^2}$.

Applying the formulas and the rules of differentiation, find the derivatives of the following functions:

741. $y = 2x^3 - 5x^2 + 7x + 4$.
Solution. $y' = (2x^3)' - (5x^2)' + (7x)' + (4)' = 2(x^3)' - 5(x^2)' + 7x' + 4' =$

$$= 2 \cdot 3x^2 - 5 \cdot 2x + 7 \cdot 1 + 0 = 6x^2 - 10x + 7.$$

742. $y = x^2 \cdot e^x$.
Solution. $y' = x^2(e^x)' + e^x(x^2)' = x^2e^x + 2xe^x = xe^x(x + 2)$.

743. $y = x^3 \text{ arc tan } x$.
Solution. $y' = x^3 (\arctan x)' + \arctan x \cdot (x^3)' = x^3 \cdot \dfrac{1}{1 + x^2} +$

$$+ 3x^2 \cdot \arctan x = \frac{x^3}{1 + x^2} + 3x^2 \cdot \arctan x.$$

744. $y = x\sqrt{x}\,(3 \ln x - 2)$.
Solution. Let us rewrite the given function in the form $y = x^{3/2} \cdot (3 \ln x - 2)$. Then

$$y' = x^{3/2} \cdot \frac{3}{x} + \frac{3}{2}\, x^{1/2}(3\ln x - 2) = 3x^{1/2} + \frac{9}{2}\, x^{1/2} \cdot \ln x - 3x^{1/2} = \frac{9}{2}\, \sqrt{x}\ln x.$$

745. $y = \dfrac{\arcsin y}{x}$.

Solution. $y' = \dfrac{x(\arcsin x)' - \arcsin x(x)'}{x^2} = \dfrac{x\dfrac{1}{\sqrt{1 - x^2}} - \arcsin x}{x^2}$

$$= \frac{x - \sqrt{1 - x^2} \cdot \arcsin x}{x^2\sqrt{1 - x^2}}.$$

746. $y = \dfrac{\sin x - \cos x}{\sin x + \cos x}$.

Solution.

$$y' = \frac{(\sin x + \cos x)(\cos x + \sin x) - (\sin x - \cos x)(\cos x - \sin x)}{(\sin x + \cos x)^2} =$$

$$= \frac{(\sin x + \cos x)^2 + (\sin x - \cos x)^2}{(\sin x + \cos x)^2} = \frac{2}{(\sin x + \cos x)^2}.$$

747. $y = (2x^3 + 5)^4$.

Solution. Let us denote $2x^3 + 5 = u$; then $y = u^4$. By the rule of differentiating a composite function we have

$$y' = (u^4)'_u \cdot (2x^3 + 5)'_x = 4u^3(6x^2) = 24x^2(2x^3 + 5)^3.$$

748. $y = \tan^6 x$.

Solution. $y' = 6\tan^5 x \cdot (\tan x)' = 6\tan^5 x \cdot \sec^2 x$.

749. $y = \cos^2 x$.

Solution. $y' = 2\cos x(\cos x)' = -2\cos x \cdot \sin x = -\sin 2x$.

750. $y = \sin(2x + 3)$.

Solution. $y' = \cos(2x + 3) \cdot (2x + 3)' = 2\cos(2x + 3)$.

751. $y = \tan \ln x$.

Solution. $y' = \sec^2 \ln x' \cdot (\ln x)' = \frac{1}{x} \cdot \sec^2 \ln x$.

752. $y = \sin^3 \frac{x}{3}$.

Solution. $y' = 3\sin^2\frac{x}{3} \cdot \left(\sin\frac{x}{3}\right)' = 3\sin^2\frac{x}{3} \cdot \cos\frac{x}{3}\left(\frac{x}{3}\right)' = \sin^2\frac{x}{3} \cdot \cos\frac{x}{3}$.

753. $y = \ln(x^2 + 5)$.

Solution. $y' = \dfrac{1}{x^2 + 5}(x^2 + 5)' = \dfrac{2x}{x^2 + 5}$.

754. $y = \ln \tan \frac{x}{2}$.

Solution. $y' = \dfrac{1}{\tan(x/2)} \cdot (\tan(x/2))' = \dfrac{1}{\tan(x/2)} \cdot \sec^2(x/2) \cdot (x/2)' =$

$$= \frac{1}{2\tan(x/2)\cos^2(x/2)} = \frac{1}{2\sin(x/2)\cos(x/2)} = \frac{1}{\sin x}.$$

755. $y = \ln(x + \sqrt{x^2 + 1})$.

Solution. $y' = \dfrac{1}{x + \sqrt{x^2 + 1}} \cdot (x + \sqrt{x^2 + 1})' = \dfrac{1}{x + \sqrt{x^2 + 1}} \times$

$$\times\left(1 + \frac{2x}{2\sqrt{x^2 + 1}}\right) = \frac{1}{x + \sqrt{x^2 + 1}} \cdot \frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}} = \frac{1}{\sqrt{x^2 + 1}}.$$

756. $y = \ln(\sqrt{2\sin x + 1} + \sqrt{2\sin x - 1})$.

Solution. $y' = \left(\dfrac{1}{\sqrt{2\sin x + 1} + \sqrt{2\sin x - 1}}\right) \cdot \left(\sqrt{2\sin y + 1} + \right.$

$$\left. + \sqrt{2\sin x - 1}\right)' =$$

$$= \frac{1}{\sqrt{2\sin x + 1} + \sqrt{2\sin x - 1}}\left(\frac{2\cos x}{2\sqrt{2\sin x + 1}} + \frac{2\cos x}{2\sqrt{2\sin x - 1}}\right) =$$

$$= \frac{1}{\sqrt{2\sin x + 1} + \sqrt{2\sin x - 1}} \frac{\cos x(\sqrt{2\sin x + 1} + \sqrt{2\sin x - 1})}{\sqrt{4\sin^2 x - 1}} =$$

$$= \frac{\cos x}{\sqrt{4\sin^2 x - 1}}.$$

757. $y = \dfrac{x}{2}\sqrt{x^2 + k} + \dfrac{k}{2} \cdot \ln\left(x + \sqrt{x^2 + k}\right)$.

Solution. $y' = \dfrac{x}{2} \cdot \dfrac{2x}{2\sqrt{x^2 + k}} + \dfrac{1}{2} \cdot \sqrt{x^2 + k} + \dfrac{k}{2} \cdot \dfrac{1}{x + \sqrt{x^2 + k}}$

$$\left(1 + \frac{2x}{2\sqrt{x^2 + k}}\right) = \frac{x^2}{2\sqrt{x^2 + k}} + \frac{\sqrt{x^2 + k}}{2} + \frac{k}{2} \times$$

$$\times \frac{1}{x + \sqrt{x^2 + k}} \cdot \frac{\sqrt{x^2 + k} + x}{\sqrt{x^2 + k}} = \frac{x^2 + k}{\sqrt{x^2 + k}} = \sqrt{x^2 + k}.$$

758. $y = \arcsin \dfrac{2x^2}{1 + x^4}$, $|x| < 1$.

Solution.

$$y' = \frac{1}{\sqrt{1 - \left(\dfrac{2x^2}{1 + x^4}\right)^2}} \cdot \left(\frac{2x^2}{1 + x^4}\right)' = \frac{1}{\sqrt{1 - \left(\dfrac{2x^2}{1 + x^4}\right)^2}} \times$$

$$\times \frac{(1 + x^4) \cdot 4x - 2x^2 \cdot 4x^3}{(1 + x^4)^2} = \frac{1}{\sqrt{1 - 2x^4 + x^8}} \cdot \frac{4x(1 - x^4)}{1 + x^4} = \frac{4x}{1 + x^4}.$$

759. $y = \arctan \dfrac{\ln x}{3}$.

Solution. $y' = \dfrac{1}{1 + (\ln^2 x)/9} \cdot \dfrac{1}{3x} = \dfrac{3}{x(9 + \ln^2 x)}$.

760. $y = e^x \cdot \arctan e^x - \ln \sqrt{1 + e^{2x}}$.
Solution. Writing the given function in the form

$$y = e^x \arctan e^x - \frac{1}{2} \ln(1 + e^{2x}),$$

we obtain

$$y' = e^x \cdot \frac{1}{1 + e^{2x}} \cdot e^x + e^x \arctan e^x - \frac{1}{2} \frac{1}{1 + e^{2x}} \cdot e^{2x} \cdot 2 =$$

$$= \frac{e^{2x}}{1 + e^{2x}} + e^x \cdot \arctan e^x - \frac{e^{2x}}{1 + e^{2x}} = e^x \arctan e^x.$$

761. $y = \dfrac{\sin x}{\cos^2 x} + \ln \dfrac{1 + \sin x}{\cos x}$.

Solution. Let us transform the given function:

$$y = \frac{\sin x}{\cos^2 x} + \ln(1 + \sin x) - \ln \cos x.$$

Then we have

$$y' = \frac{\cos^2 x \cos x - \sin x \cdot 2 \cos x(-\sin x)}{\cos^4 x} +$$

$$+ \frac{1}{1 + \sin x} \cos x - \frac{1}{\cos x}(-\sin x),$$

or

$$y' = \frac{\cos^2 x + 2 \sin^2 x}{\cos^3 x} + \frac{\cos x(1 - \sin x)}{1 - \sin^2 x} + \frac{\sin x}{\cos x} =$$

$$= \frac{\cos^2 x + 2 \sin^2 x}{\cos^3 x} + \frac{1 - \sin x}{\cos x} + \frac{\sin x}{\cos x} =$$

$$= \frac{\cos^2 x + 2\sin^2 x}{\cos^3 x} + \frac{1}{\cos x} = \frac{2}{\cos^3 x} = 2\sec^3 x.$$

762. $y = \dfrac{1}{2} \tan^2 \sqrt{x} + \ln \cos \sqrt{x}$.

Solution. $y' = \tan\sqrt{x}\cdot\sec^2\sqrt{x}\,\frac{1}{2\sqrt{x}} + \frac{1}{\cos\sqrt{x}}\cdot(-\sin\sqrt{x})\,\frac{1}{2\sqrt{x}} =$

$$= \frac{1}{2\sqrt{x}}\tan\sqrt{x}\,(\sec^2\sqrt{x}-1) = \frac{1}{2\sqrt{x}}\tan^3\sqrt{x}.$$

763. $y = 5\sinh^3\frac{x}{15} + 3\sinh^5\frac{x}{15}$.

Solution. We find

$$y' = 15\sinh^2\frac{x}{15}\cdot\cosh\frac{x}{15}\cdot\frac{1}{15} + 15\sinh^4\frac{x}{15}\cdot\cosh\frac{x}{15}\cdot\frac{1}{15} =$$

$$= \sinh^2\frac{x}{15}\cdot\cosh\frac{x}{15}\left(1+\sinh^2\frac{x}{15}\right),$$

whence, using the relation $\cosh^2 x - \sinh^2 x = 1$, we finally get $y' = \sinh^2\frac{x}{15}\,.\cosh^3\frac{x}{15}$.

764. $y = x^{x^2}$.

Solution. Here the base and the exponent depend on x. Taking the logarithm, we obtain

$$\ln y = x^2\ln x.$$

Let us differentiate both sides of the last equality with respect to x. Since y is a function of x, $\ln y$ is a composite function of x and $(\ln y)' = \frac{1}{y}\cdot y'$. Consequently,

$$\frac{y'}{y} = x^2\cdot\frac{1}{x} + 2x\cdot\ln x, \qquad \frac{y'}{y} = x(1+2\ln x),$$

i.e.

$$y' = xy(1+2\ln x) = xx^{x^2}(1+2\ln x) = x^{x^2+1}(1+2\ln x).$$

765. $y = (\sin x)^{\tan x}$.

Solution. We have $\ln y = \tan x\cdot\ln\sin x$, whence we get

$$\frac{y'}{y} = \tan x\cdot\frac{1}{\sin x}\cdot\cos x + \sec^2 x\cdot\ln\sin x = 1 + \sec^2 x\cdot\ln\sin x;$$

$$y' = y(1+\sec^2 x\cdot\ln\sin x) = (\sin x)^{\tan x}(1+\sec^2 x\cdot\ln\sin x).$$

766. $y\dfrac{(2x-1)^3\cdot\sqrt{3x+2}}{(5x+4)\sqrt[3]{1-x}}$.

Solution. Before seeking the derivative, it is useful to take the logarithm of the given function:

$$\ln y = 3\ln(2x-1) + \frac{1}{2}\ln(3x+2) - 2\ln(5x+4) - \frac{1}{3}\ln(1-x);$$

$$\frac{y'}{y} = \frac{3}{2x-1}\cdot 2 + \frac{1}{2}\cdot\frac{3}{3x+2} - 2\cdot\frac{5}{5x+4} + \frac{1}{3(1-x)};$$

$$y' = \frac{(2x-1)^3\sqrt{3x+2}}{(5x+4)^2\sqrt[3]{1-x}}\left[\frac{6}{2x-1} + \frac{3}{2(3x+2)} - \frac{10}{5x+4} + \frac{1}{3(1-x)}\right].$$

Find the derivatives of the following functions:

767. $y = \dfrac{7}{x^3}$. **768.** $y = \dfrac{3}{4}x\sqrt[3]{x}$.

769. $y = \dfrac{2}{7}x^3\sqrt{x} - \dfrac{4}{11}x^5\sqrt{x} + \dfrac{2}{15}x^7\sqrt{x}$.

770. $y = (x^2+2x+2)e^{-x}$. **771.** $y = 3x^3\ln x - x^3$.

772. $y = \dfrac{2^{3x}}{3^{2x}}$. **773.** $y = x^2\sin x + 2x\cos x - 2\sin x$.

774. $y = \ln(2x^3+3x^2)$. **775.** $y = \sqrt{1-3x^2}$.

776. $y = x\arccos\dfrac{x}{2} - \sqrt{4-x^2}$.

777. $y = \sqrt{x}\arcsin\sqrt{x} + \sqrt{1-x}$.

778. $y = \left(\sin\dfrac{x}{2} - \cos\dfrac{x}{2}\right)^2$. **779.** $y = \cos^3(x/3)$.

780. $y = \ln\tan\dfrac{2x+1}{4}$. **781.** $y = \ln\sqrt{\dfrac{1+\sin x}{1-\sin x}}$.

782. $y = \tan 2x + \dfrac{2}{3}\tan^3 2x + \dfrac{1}{5}\tan^5 2x$.

783. $y = \dfrac{1}{3}\sin^3\sqrt{x} - \dfrac{2}{5}\sin^5\sqrt{x} + \dfrac{1}{7}\sin^7\sqrt{x}$.

784. $y = \ln(3x^2+\sqrt{9x^4+1})$. **785.** $y = \dfrac{x}{2}\sqrt{a^2-x^2} + \dfrac{a^2}{2}\arcsin\dfrac{x}{a}$.

786. $y = \ln\dfrac{\sqrt{4\tan x+1} - 2\sqrt{\tan x}}{\sqrt{4\tan x+1} + 2\sqrt{\tan x}}$. **787.** $y = -\cot^2\dfrac{x}{2} - 2\ln\sin\dfrac{x}{2}$.

788. $y = \arctan\sqrt{4x^2-1}$. **789.** $y = \arctan\dfrac{x}{\sqrt{a^2-x^2}}$.

790. $y = \arctan \dfrac{1 - \sqrt{1 - x^2}}{x}$. **791.** $y = \arcsin \dfrac{2x^3}{1 + x^6}$, if $|x| < 1$.

792. $y = \arccos \dfrac{9 - x^2}{9 + x^2}$. **793.** $y = e^{-x} - \sin e^{-x} \cos e^{-x}$.

794. $y = \arctan \sqrt{\dfrac{1 - x}{1 + x}}$. **795.** $y = \ln \dfrac{(x - 1)(x - 3)^3}{(x - 2)^3(x - 4)}$.

796. $y = 1 - e^{\sin^2 3x} \cos^2 3x$. **797.** $y = \ln \dfrac{2 \ln^2 \sin x + 3}{2 \ln^2 \sin x - 3}$.

798. $y = \ln (\sec x + \tan x)$. **799.** $y = -\ln (\operatorname{cosec} x + \cot x)$.

800. $y = e^{\sqrt{2x}}(\sqrt{2x} - 1)$. **801.** $y = \ln \dfrac{x^5}{x^5 + 2}$.

802. $y = \left(\dfrac{\sin x}{1 + \cos x}\right)^2$. **803.** $y = \arcsin \dfrac{\sin x}{\sqrt{1 + \sin^2 x}}$.

804. $y = -\operatorname{cosec}^2 (x/2)$. **805.** $y = \sin (\ln x) \cdot \cos (\ln x) - \ln (1/x)$.

806. $y = (x^5 + 3)[\ln (x^5 + 3) - 1]$. **807.** $y = \arcsin \sqrt{1 - 0.2x^2}$.

808. $y = 0.5[(x + \alpha)\sqrt{x^2 + 2\alpha x + \beta} + (\beta - \alpha^2) \ln (x + \alpha + \sqrt{x^2 + 2\alpha x + \beta})]$.

809. $y = \arcsin e^x + \arcsin \sqrt{1 - e^{2x}}$.

810. $y = m\sqrt{x^2 + 2\alpha x + \beta} + (n - m\alpha) \ln (x + \alpha + \sqrt{x^2 + 2\alpha x + \beta})$.

811. $y = \dfrac{x}{\sqrt{1 - mx^2}}$. **812.** $y = x^2 + 2x \sin x \cos x + \cos^2 x$.

813. $y = \cot x \operatorname{cosec} x + \ln (\cot x + \operatorname{cosec} x)$. **814.** $y = \dfrac{\sin x}{1 + \ln \sin x}$.

815. $y = 3x \sin^3 x + 3 \cos x - \cos^3 x$. **816.** $y = \ln \dfrac{\sqrt{1 + x^2} - 1}{\sqrt{1 + x^2} + 1}$.

817. $y = e^x - \sin e^x \cos^3 e^x - \sin^3 e^x \cos e^x$.

818. $y = \arctan (x + 1) + \dfrac{x + 1}{x^2 + 2x + 2}$.

819. $y = x(\ln^3 x - 3 \ln^2 x + 6 \ln x - 6)$.

820. $y = \ln \sin \sqrt{x} \tan \sqrt{x} - \sqrt{x}$. **821.** $y = \arctan \dfrac{x^x - x^{-x}}{2}$.

822. $y = \dfrac{1}{64}\left(\tan^4 \dfrac{x}{2} - \cot^4 \dfrac{x}{2}\right) + \dfrac{1}{8}\left(\tan^2 \dfrac{x}{2} - \cot^2 \dfrac{x}{2}\right) + \dfrac{3}{8} \ln \tan \dfrac{x}{2}$.

823. $y = \ln \tan \dfrac{x}{2} + \cos x + \dfrac{1}{3} \cos^3 x$. **824.** $y = -\dfrac{2 \cos (x/2)}{\sin (x/2) + 3 \cos (x/2)}$.

825. $y = \dfrac{1}{2} \tan^2 \sin x + \ln \cos \sin x$. **826.** $y = \ln \left(1 - \dfrac{1}{x}\right) + \dfrac{1}{x}$.

827. $y = \ln \dfrac{\sqrt{x^2 + 2x}}{x + 1}$. **828.** $y = 2x \tan 2x + \ln \cos 2x - 2x^2$.

829. $y = \arccos (2e^{2x} - 1)$. **830.** $y = \ln \ln x(\ln \ln \ln x - 1)$.

831. $y = \dfrac{x - e^{2x}}{x + e^{2x}}$. **832.** $y = \ln \dfrac{x \ln x - 1}{x \ln x + 1}$.

833. $y = \arctan \dfrac{3x - x^3}{1 - 3x^2}$. **834.** $y = \ln \tan \dfrac{e^{2 \sin x}}{4}$.

835. $y = \dfrac{a}{2} \sin^2 x + \dfrac{b}{2} \cos^2 x - \dfrac{a + b}{4} \cos 2x$.

836. $y = \tan^3 \tan x + 3 \tan \tan x$. **837.** $y = \dfrac{\arctan x}{x} - \ln \dfrac{x}{\sqrt{1 + x^2}}$.

838. $y = \dfrac{\ln x}{x^5} + \dfrac{1}{5x^5}$. **839.** $y = \sqrt{2x + 1}\,[\ln (2x + 1) - 2]$.

840. $y = \sec x(1 + \ln \cos x)$. **841.** $y = e^x\sqrt{1 - e^{2x}} - \arcsin e^x$.

842. $y = 2^{\cos^3 x - 3 \cos x}$. **843.** $y = \dfrac{e^x \cdot 2^{5x}}{3^{4x}}$.

844. $y = \dfrac{x + 1}{x} - e^{-\ln \frac{x}{x+1}}$. **845.** $y = x \sin x \cos x + \dfrac{1}{2} \cos^2 x$.

846. $y = \dfrac{x^2 e^{x^2}}{x^2 + 1}$. **847.** $y = \ln \tan \dfrac{x}{2} - \dfrac{x}{\sin x}$.

848. $y = 2(\tan \sqrt{x} - \sqrt{x})$. **849.** $y = \dfrac{1}{4a} \ln \dfrac{x - a}{x + a} + \dfrac{1}{2a} \arctan \dfrac{x}{a}$.

850. $y = \ln \dfrac{\sqrt{x^4 + 1} - x^2}{\sqrt{x^4 + 1} + x^2}$. **851.** $y = e^{0.5 \tan^2 x} \cos x$.

852. $y = \arctan \dfrac{2x^4}{1 - x^8}$. **853.** $y = x^2 e^{x^2} \ln x$.

854. $y = \arccos \sqrt{1 - 2^x}$. **855.** $y = \log_{x^2} 2$.

856. $y = -m\sqrt{-x^2 + 2\alpha x + \beta} + (m\alpha + n) \arcsin \dfrac{x - \alpha}{\sqrt{\alpha + \beta}}$.

857. $y = \log_2 \sin^2 x$. **858.** $y = \log_a (x + \sqrt{x^2 + 9})$.

859. $y = x^{\arcsin x}$. **860.** $y = \left(\dfrac{x - 1}{x + 1}\right)^4$. **861.** $y = \dfrac{2^x(x + 1)^3}{(x - 1)^2\sqrt{2x + 1}}$.

862. $y = \ln \left(\dfrac{1}{x} + \sqrt{\dfrac{1}{x^2} + 1}\right)$.

863. $y = \dfrac{x^{m+1}}{(m + 1)^2 + n^2} [(m + 1) \cos (n \ln x) + n \sin (n \ln x)]$.

864. $y = (x \tan x + \ln \cos x) \cdot \tan (x \tan x + \ln \cos x) + \ln \cos (x \tan x + \ln \cos x)$.

865. $y = (x \cos x - \sin x)[\ln (x \cos x - \sin x) - 1]$.

866. $y = 3 \sin (xe^x - e^x) - \sin^3 (xe^x - e^x)$.

867. $y = \arccos (2x\sqrt{1 - x^2})$. **868.** $y = |x|\ (x \neq 0)$.

869. $y = |f(x)|$. **870.** $y = |3x - 5|$.
871. $y = e^{|x|}$. **872.** $y = |x| + |x - 2|$.
873. $y = xe^x(\sin x - \cos x) + e^x \cos x$.
874. $y = \ln[x \sin x + \cos x + \sqrt{(x \sin x + \cos x)^2 + 1}]$.
875. $y = \frac{x^x}{e^x}(x \ln x - x - 1)$. **876.** $y = \log_{\cos x} \sin x$.
877. $y = \log_{e^2}(x^n + \sqrt{x^{2n} + 1})$. **878.** $y = \log_x e$.
879. $y = \log_{x^2} x$. **880.** $y = \log_{x^2} x^x$. **881.** $y = x^{1/\ln x}$.
882. $y = x^x$. **883.** $y = x^{-x} \cdot 2^x \cdot x^2$.
884. $y = x^{\ln x}$. **885.** $y = \dfrac{x^2\sqrt{x+1}}{(x-1)^3\sqrt[5]{5x-1}}$.
886. Show that $(\sec x)' = \sec x \tan x$.
887. Show that $(\operatorname{cosec} x)' = -\operatorname{cosec} x \cot x$.
888. Show that $(u^v)' = vu^{v-1} \cdot u' + u^v \cdot v' \ln u$.
889. Derive the formulas for differentiating arcsec x and arccosec x.
890. What is the expression $u = y^2 + y'^2 + 4y^2/y'^2$ equal to if $y = 2 \cos x$?
891. Show that the function $y = (x^2 + 1)(e^x + C)$ turns the equation $y' - \frac{2xy}{x^2 + 1} = e^x(x^2 + 1)$ into an identity.

7.1.2. **Differentiation of implicit functions.** Assume that the equation $F(x, y) = 0$ specifies y as an implicit function of x. In what follows we shall consider this function to be differentiable.

Differentiating both sides of the equation $F(x, y) = 0$ with respect to x, we obtain a first-degree equation with respect to y'. This equation easily yields y', that is, the derivative of the implicit function.

892. Find the derivative y'_x from the equation $x^2 + y^2 = 4$.

Solution. Since y is a function of x, we shall treat y^2 as a composite function of x. Consequently, $(y^2)' = 2yy'$. Differentiating both sides of the given equation with respect to x, we get $2x + 2yy' = 0$, i.e. $y' = -x/y$.

893. Find the derivative y'_x from the equation $x^3 + \ln y - x^2e^y = 0$.

Solution. Differentiating both sides of the equation with respect to x, we obtain

$$3x^2 + \frac{y'}{y} - x^2e^yy' - 2xe^y = 0, \quad \text{i.e.} \quad y' = \frac{(2xye^y - 3x^2)y}{1 - x^2ye^y}.$$

Find the derivative y'_x of the following implicit functions.

894. $x^3 + y^3 - 3xy = 0$.
895. $Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$.
896. $x^4 - 6x^2y^2 + 9y^4 - 5x^2 + 15y^2 - 100 = 0$.
897. $x^y - y^x = 0$. **898.** $x \sin y + y \sin x = 0$.
899. $e^x + e^y - 2^{xy} - 1 = 0$.
900. $\sin(y - x^2) - \ln(y - x^2) + 2\sqrt{y - x^2} - 3 = 0$.

901. $\frac{y}{x} + e^{y/x} - \sqrt[3]{\frac{y}{x}} = 0.$ **902.** $x^{y^2} + y^2 \ln x - 4 = 0.$

903. $x^2 \sin y + y^3 \cos x - 2x - 3y + 1 = 0.$

7.1.3. Differentiation of functions represented parametrically. If the function y of the argument x is specified by the parametric equations $x = \varphi(t)$, $y = \psi(t)$, then

$$y'_x = \frac{y'_t}{x'_t}, \quad \text{or} \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

904. Find $y' = \frac{dy}{dx}$ if $x = t^3 + 3t + 1$, $y = 3t^5 + 5t^3 + 1$.

Solution. We find $\frac{dx}{dt} = 3t^2 + 3$, $\frac{dy}{dt} = 15t^4 + 15t^2$. Consequently, $\frac{dy}{dx} = \frac{15t^4 + 15t^2}{3t^2 + 3} = 5t^2$.

905. Find $y' = \frac{dy}{dx}$ if $x = a \cos t$, $y = a \sin t$.

906. Find $y' = \frac{dy}{dx}$ if $x = e^{-t} \sin t$, $y = e^t \cos t$.

907. Find $\rho' = \frac{d\rho}{d\theta}$ if $\rho = \left(\frac{2}{3}\sqrt{\alpha} + 1\right)\alpha$, $\theta = \sqrt{\alpha}e^{\sqrt{\alpha}}$.

908. Find $y' = \frac{dy}{dx}$ if $x = \cosh t$, $y = \sinh t$.

7.1.4. Application of the derivative to problems of geometry and mechanics. If a curve is specified by the equation $y = f(x)$, then $f'(x_0) = \tan \alpha$, where α is the angle formed by the positive direction of the Ox axis and the tangent to the curve at the point with the abscissa x_0.

The *equation of the tangent* to the curve $y = f(x)$ at the point $M_0(x_0; y_0)$ has the form

$$y - y_0 = y'_0(x - x_0),$$

where y'_0 is the value of the derivative y' at the point $M_0(x_0, y_0)$.

A *normal* to a curve is a straight line which is perpendicular to the tangent and passes through the point of tangency.

The *equation of a normal* has the form

$$y - y_0 = -\frac{1}{y'_0}(x - x_0).$$

The *angle between two curves* $y = f_1(x)$ and $y = f_2(x)$ at the point of their intersection $M_0(x_0; y_0)$ is the angle between the tangents to those curves at the point M_0. The angle can be found by the formula

$$\tan \varphi = \frac{f_2'(x_0) - f_1'(x_0)}{1 + f_1'(x_0) \cdot f_2'(x_0)} .$$

If a law of motion $s = s(t)$ is given governing the rectilinear motion of a point, then the rate of motion at the moment t_0 is the derivative of the path with respect to time: $v = s'(t_0)$.

909. What angle is formed between the abscissa axis and the tangent to the curve $y = (2/3)x^5 - (1/9)x^3$ drawn at the point with the abscissa $x = 1$?

Solution. We find the derivative $y' = (10/3)x^4 - (1/3)x^2$; at $x = 1$ we have $y' = 3$, i.e. $\tan \alpha = 3$, whence $\alpha = \arctan 3 \approx 71°34'$.

910. What angle is formed between the abscissa axis and the tangent to the parabola $y = x^2 - 3x + 5$ drawn at the point $M(2; 3)$? Write the equation of the tangent.

911. Derive the equations of the tangent and the normal to the curve $x^2 + 2xy^2 + 3y^4 = 6$ at the point $M(1; -1)$.

Solution. We find the derivative from the equation of the curve:

$$2x + 2y^2 + 4xyy' + 12y^3y' = 0, \quad \text{i.e.} \quad y' = -\frac{x + y^2}{2xy + 6y^3} .$$

Consequently, $y_0' = -\dfrac{1 + (-1)^2}{2 \cdot 1(-1) + 6(-1)^3} = \dfrac{1}{4}$.

The equation of the tangent is

$$y + 1 = \frac{1}{4}(x - 1), \quad \text{or} \quad x - 4y - 5 = 0.$$

The equation of the normal is

$$y + 1 = -4(x - 1), \quad \text{or} \quad 4x + y - 3 = 0.$$

912. Find the angle between the parabolas $y = 8 - x^2$ and $y = x^2$.

Solution. Solving simultaneously the equations of the parabolas, we find the points of their intersection $A(2; 4)$ and $B(-2; 4)$. Next we differentiate the equations of the parabolas: $y' = -2x$, $y' = 2x$. Then we find the slopes of the tangents to the parabolas at the point A (that is, the values of the derivatives at $x = 2$): $k_1 = -4$, $k_2 = 4$. It follows that $\tan \varphi_1 = \dfrac{4 + 4}{1 - 16} = -\dfrac{8}{15}$, $\varphi_1 = \arctan(-8/15)$. The angle between the curves at the point B can be determined in a similar way: $\varphi_2 = \arctan(8/15)$.

913. Find the equation of the normal to the parabola $y^2 = 2px$ at the point $M(x_0, y_0)$.

914. Derive the equation of the tangent to the hyperbola $x^2/9 - y^2/8 = 1$ drawn at the point $M(-9, -8)$.

915. Derive the equations of the tangent and the normal to the astroid $x = \sqrt{2}\cos^3 t$, $y = \sqrt{2}\sin^3 t$ drawn at the point for which $t = \pi/4$.

916. Derive the equations of the tangent and the normal to the cycloid $x = t - \sin t$, $y = 1 - \cos t$ drawn at the point for which $t = \pi/2$.

917. Derive the equations of the tangent and the normal to the semicubical parabola $x = t^2$, $y = t^3$ drawn at the point for which $t = 2$.

918. Show that the equation of the tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point $M(x_0; y_0)$ has the form $xx_0/a^2 + yy_0/b^2 = 1$.

919. What angle is formed between the abscissa axis and the tangent to the curve $y = \sinh x$ drawn at the point $(0; 0)$?

920. Derive the equations of the tangent and the normal to the catenary $y = \cosh(x/2)$ at the point where $x = 2\ln 2$.

921. Derive the equation of the tangent to the equilateral hyperbola $x = \cosh t$, $y = \sinh t$ at the point $t = t_0$.

922. Find the angle between the curve $y = x - x^3$ and the straight line $y = 5x$.

923. Find the angle between the curves $y = x^3$ and $y = 1/x^2$.

924. Find the angle between the lines $y = 1 + \sin x$ and $y = 1$.

925. Find the angle between the curves $x^2 + y^2 = 5$ and $y^2 = 4x$.

926. Find the angle between the curves $y = \sqrt{2}\sin x$, $y = \sqrt{2}\cos x$.

927. For the rectilinear motion of a point the dependence of the path on time is given by the equation $s = t^5/5 + (2/\pi)\sin(\pi t/8)$ (t is in seconds, s in metres). Determine the speed of motion at the end of the 2nd second.

Solution. We find the derivative of the path with respect to time:

$$\frac{ds}{dt} = t^4 + \frac{1}{4}\cos\frac{\pi t}{8}.$$

At $t = 2$ we have $\dfrac{ds}{dt} = 16 + \dfrac{1}{8}\sqrt{2} \approx 16.18$. Consequently, $v \approx 16.18$ m/s.

928. A point is moving along the parabola $y = x(8 - x)$ so that its abscissa varies with time t according to the law $x = t\sqrt{t}$ (t is in seconds, x in metres). What is the rate of change of the ordinate at the point $M(1; 7)$?

Solution. Let us find the law of variation of the ordinate; substituting $t\sqrt{t}$ for x in the equation of the parabola, we obtain $y = 8t\sqrt{t} - t^3$. The rate of change of the ordinate is the derivative of the ordinate with respect to time: $y'_t = 12\sqrt{t} - 3t^2$. For the point $M(1; 7)$ the value of t is 1. Consequently, $y'_{t=1} = 9$, that is, the rate of change of the ordinate is 9 m/s.

929. The dependence of the path on time is given by the equation $s = t\ln(t + 1)$ (t is in seconds, s in metres). Find the speed of motion at the end of the 2nd second.

930. A point is moving along the cubical parabola $y = x^3$ so that its ordinate varies with time t according to the law $y = at^3$. At what rate does the abscissa change with time?

7.1.5. Derivatives of higher orders. The *derivative of the second order* (the *second derivative*) of the function $y = f(x)$ is the derivative of its derivative. The second derivative is designated as y'', or d^2y/dx^2, or $f''(x)$.

If $s = f(x)$ is the law of the rectilinear motion of a point, then the second derivative of the path with respect to time d^2s/dt^2 is the acceleration of that motion.

Similarly, the *third derivative* of the function $y = f(x)$ is the derivative of the second-order derivative: $y''' = (y'')'$.

In general, the *nth-order derivative* of the function $y = f(x)$ is the derivative of the $(n - 1)$th-order derivative: $y^{(n)} = (y^{(n-1)})'$. The nth-order derivative is designated as $y^{(n)}$, or d^ny/dx^n or $f^{(n)}(x)$.

Derivatives of higher orders (the second, the third, etc.) can be calculated by successive differentiation of the given function.

If the function is represented parametrically

$$x = \varphi(t), \quad y = \psi(t),$$

then the derivatives y'_x, y''_{xx}, . . ., can be calculated by the formulas

$$y'_x = \frac{y'_t}{x'_t}, \quad y''_{xx} = \frac{(y'_x)'_t}{x'_t}, \quad y'''_{xxx} = \frac{(y''_{xx})'_t}{x'_t}, \quad \text{etc.}$$

The second-order derivative can also be calculated by the formula

$$y''_{xx} = \frac{y''_{tt}x'_t - x''_{tt}y'_t}{(x'_t)^3}.$$

931. $y = x^5 + 2x^4 - 3x^3 - x^2 - \frac{1}{2}x + 7$. Find y', y'', y''', ...

Solution. We have

$$y' = 5x^4 + 8x^3 - 9x^2 - 2x - \frac{1}{2},$$

$$y'' = 20x^3 + 24x^2 - 18x - 2,$$

$$y''' = 60x^2 + 48x - 18,$$

$$y^{IV} = 120x + 48, \quad y^{V} = 120, \quad y^{VI} = y^{VII} = \ldots = 0$$

932. $y = \ln x$. Find $y^{(n)}$.

Solution. We have

$$y' = \frac{1}{x} = x^{-1}.$$

$$y'' = -1 \cdot x^{-2},$$

$$y''' = 1 \cdot 2x^{-3},$$

$$y^{IV} = -1 \cdot 2 \cdot 3x^{-4},$$

....

$$y^{(n)} = 1 \cdot 2 \cdot 3 \ldots (n-1)(-1)^{n-1} \cdot x^{-n} = (-1)^{n-1}\frac{(n-1)!}{x^n}.$$

933. $y = 2^x$. Find $y^{(n)}$.

Solution. We have $y' = 2^x \cdot \ln 2$, $y'' = 2^x \cdot \ln^2 2$, $y''' = 2^x \cdot \ln^3 2, \ldots, y^{(n)} = 2^x \ln^n 2$.

934. $y = \sin x$. Find $y^{(n)}$.

Solution. We have

$$y' = \cos x = \sin\left(x + \frac{\pi}{2}\right),$$

$$y'' = -\sin x = \sin\left(x + 2 \cdot \frac{\pi}{2}\right),$$

$$y''' = -\cos x = \sin\left(x + 3 \cdot \frac{\pi}{2}\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y^{(n)} = \sin\left(x + n \cdot \frac{\pi}{2}\right).$$

935. Find $y' = \dfrac{dy}{dx}$, $y'' = \dfrac{d^2y}{dx^2}$, if $x = a\cos^3 t$, $y = a\sin^3 t$.

Solution. We have

$$y' = \frac{dy}{dx} = \frac{(a\sin^3 t)'_t}{(a\cos^3 t)'_t} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t;$$

$$y'' = \frac{d^2y}{dx^2} = \frac{(-\tan t)'_t}{(a\cos^3 t)'_t} = \frac{-\sec^2 t}{-3a\cos^2 t\sin t} = \frac{1}{3a\sin t\cos^4 t}.$$

Find the second-order derivatives:

936. $y = -\dfrac{22}{x+5}$. **937.** $y = \dfrac{1}{4}x^2(2\ln x - 3)$.

938. $y = \dfrac{1}{3}x^2\sqrt{1-x^2} + \dfrac{2}{3}\sqrt{1-x^2} + x\arcsin x$.

939. $y = -\dfrac{1}{9}x\sin 3x - \dfrac{2}{27}\cos 3x$.

940. $y = x\ln(x + \sqrt{x^2+a^2}) - \sqrt{x^2+a^2}$.

941. $\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t). \end{cases}$ **942.** $\begin{cases} x = \arccos\sqrt{t}, \\ y = \sqrt{t - t^2}. \end{cases}$

943. Show that the function $y = \sin\ln x + \cos\ln x$ satisfies the equation $x^2y'' + xy' + y = 0$.

944. Show that the function $y = x + \sin 2x$ satisfies the equation $y'' + 4y + 4x$.

945. For the rectilinear motion of a point the dependence of the path on time is specified by the equation $s = \sqrt{t}$. Find the acceleration of the point at the end of the 4th second.

Find the third-order derivatives:

946. $y = \dfrac{x}{6(x+1)}$. **947.** $y = \dfrac{1}{2}\ln^2 x$.

948. $y = (2x+3)^3\sqrt{2x+3}$. **949.** $y = \sinh^2 x$.

Hint. $\sinh 2x = 2\sinh x \cosh x$.

Find the nth-order derivatives:

950. $y = x^n\sqrt{x}$. **951.** $y = \dfrac{1}{2x+1}$. **952.** $y = 5 - 3\cos^2 x$.

953. $y = 2^x + 2^{-x}$. **954.** $y = \dfrac{ax+b}{cx+d}$. **955.** $y = e^{kx}$.

956. $y = \cos x$. **957.** $\begin{cases} x = \ln t, \\ y = 1/t. \end{cases}$ **958.** $\begin{cases} x = at + b, \\ y = \alpha t^2 + \beta t + \gamma. \end{cases}$

959. Show that the function $y = e^x + 2e^{2x}$ satisfies the equation $y''' - 6y'' + 11y' - 6y = 0$.

960. Show that the function $y = x^3$ satisfies the equation $y^{V} + y^{IV} + y''' + y'' + y' + y = x^3 + 3x^2 + 6x + 6$.

7.1.6. Differentials of the first and higher orders. The *differential* (*of the first order*) of the function $y = f(x)$ is the dominant part of its increment, linear with respect to the increment of the argument. The differential of the argument is the increment of the argument: $dx = \Delta x$.

The differential of a function is equal to the product of its derivative by the differential of the argument:

$$dy = y'dx.$$

In geometrical interpretation, the differential is the increment of the ordinate of the tangent to the graph of the function at the point $M(x; y)$ (Fig. 28).

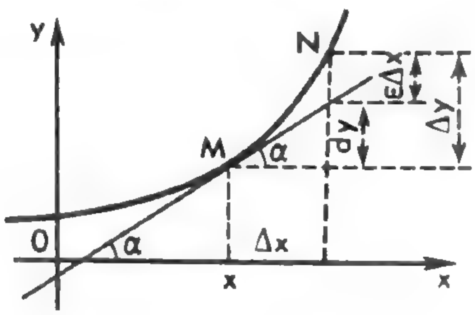

Fig. 28

The main properties of the differential

1°. $dC = 0$, where $C = \text{const}$.
2°. $d(Cu) = Cdu$.
3°. $d(u \pm v) = du \pm dv$.

4°. $d(uv) = udv + vdu$.

5°. $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$ $(v \neq 0)$.

6°. $df(u) = f'(u)du$.

If the increment Δx of the argument is small in magnitude, then

$$\Delta y \approx dy$$

and

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x.$$

Thus, we see that the differential of a function can be used in approximate calculations.

The *differential of the second order* of the function $y = f(x)$ is the differential of the first-order differential $d^2y = d(dy)$.

The third-order differential can be determined in a similar way: $d^3y = d(d^2y)$.

In general, $d^ny = d(d^{n-1}y)$.

If $y = f(x)$ and x is an independent variable, then the higher-order differentials are calculated by the formulas

$$d^2y = y''(dx)^2,\ d^3y = y'''(dx)^3,\ \dots,\ d^ny = y^{(n)}(dx)^n.$$

961. Find the differential of the function $y = \arctan x$.

Solution. $dy = (\arctan x)' \cdot dx = \frac{dx}{1 + x^2}$.

962. Find the differential of the function $s = e^{t^3}$.

Solution. $ds = e^{t^3} \cdot 3t^2dt$.

963. Find the first-, second-, and third-order differentials of the function $y = (2x - 3)^3$.

Solution. We have

$$dy = 3(2x - 3)^2 \cdot 2dx = 6(2x - 3)^2dx,$$

$$d^2y = 12(2x - 3) \cdot 2dx^2 = 24(2x - 3)dx^2,$$

$$d^3y = 24 \cdot 2dx^3 = 48dx^3.$$

964. Find the first-and second-order differentials of the function $v = e^{2t}$.

Solution. $dv = 2e^{2t}dt$, $d^2v = 4e^{2t} \cdot dt^2$.

965. Compare the increment and the differential of the function $y = 2x^3 + 5x^2$.

Solution. We find

$$\Delta y = 2(x + \Delta x)^3 + 5(x + \Delta x)^2 - 2x^3 - 5x^2 =$$

$$= (6x^2 + 10x)\Delta x + (6x + 5)\Delta x^2 + 2\Delta x^3,$$

$$dy = (6x^2 + 10x)dx.$$

The difference between the increment Δy and the differential dy is an infinitesimal of the higher order of smallness relative to Δx, and it is equal to $(6x + 5)\Delta x^2 + 2\Delta x^3$.

966. Calculate the approximate value of arcsin 0.51.

Solution. Let us consider the function $y = \arcsin x$. Putting $x = 0.5$, $\Delta x = 0.01$ and applying the formula $\arcsin(x + \Delta x) \approx \arcsin x + (\arcsin x)' \cdot \Delta x$, we obtain

$$\arcsin 0.51 \approx \arcsin 0.5 + \frac{1}{\sqrt{1-(0.5)^2}} \cdot 0.01 = \frac{\pi}{6} + 0.011 = 0.513.$$

967. Calculate the approximate value of the area of a circle whose radius is equal to 3.02 m.

Solution. We make use of the formula $S = \pi R^2$. Putting $R = 3$, $\Delta R = 0.02$, we obtain

$$\Delta S \approx dS = 2\pi R \cdot \Delta R = 2\pi \cdot 3 \cdot 0.02 = 0.12\pi.$$

Consequently, the approximate value of the area of the circle is equal to $9\pi + 0.12\pi = 9.12\pi \approx 28.66$ (sq. m).

Find the differentials of the functions:

968. $y = \frac{x}{2}\sqrt{49 - x^2} + \frac{49}{2}\arcsin\frac{x}{7}$. **969.** $x = \frac{1}{12}\ln\frac{x-6}{x+6}$.

970. $y = 2\ln\cosh(x/2)$. **971.** $y = \arctan e^{2x}$.

972. $y = x(\ln x - 1)$. Find dy, d^2y, d^3y.

973. $y = \ln(x + \sqrt{x^2 + 4})$. Find d^2y.

974. Compare the increment and the differential of the function $y = 1/x$.

975. Compute Δy and dy for the function $y = x^2 - 2x$ at $x = 3$ and $\Delta x = 0.01$.

976. Find the approximate value of arctan 1.05.

977. Find the approximate value of the volume of a sphere whose radius is equal to 2.01 m.

978. Find the approximate value of tan 46°.

979. Find the approximate value of $\ln\tan 47°15'$.

980. Find the approximate value of x from the equation $13\sin x - 15\cos x = 0$.

981. Find the approximate value of $\sqrt[4]{15.8}$.

7.2. Investigation of Functions

7.2.1. Theorems of Rolle, Lagrange, Cauchy, and Taylor's formula.

Rolle's theorem. *If the function $f(x)$ is continuous on the closed interval $[a, b]$, differentiable on the interval (a, b), and $f(a) = f(b)$, then there is at least one value $x = \xi$ on the interval (a, b) for which $f'(\xi) = 0$.*

If, in particular, $f(a) = 0$, $f(b) = 0$, then Rolle's theorem means that between two roots of the function there is at least one root of its derivative.

Lagrange's theorem (on finite increment). *If the function $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the interval (a, b) then there is at least one value $x = \xi$ on that interval for which the equality*

$$f(b) - f(a) = (b - a)f'(\xi)$$

holds true.

These theorems have the following geometrical meaning: on the arc AB of the continuous curve $y = f(x)$ having a definite tangent (nonparallel to the y-axis) at its every interior point there is at least one interior point at which the tangent is parallel to the chord AB. (For Rolle's theorem both the chord AB and the tangent are parallel to the x-axis.)

Cauchy's theorem. *If the functions $f(x)$ and $\varphi(x)$ are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) with $\varphi'(x) \neq 0$, then there is at least one value $x = \xi$ on that interval for which*

$$\frac{f(b) - f(a)}{\varphi(b) - \varphi(a)} = \frac{f'(\xi)}{\varphi'(\xi)},$$

where $a < \xi < b$.

Taylor's formula. *The function $f(x)$, $n + 1$ times differentiable on some interval containing a point a in its interior, can be represented as the sum of an nth-degree polynomial and a remainder R_n:*

$$f(x) = f(a) + \frac{f(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n,$$

$$R_n = \frac{f^{(n+1)}(\xi)}{(n + 1)!}(x - a)^{n+1},$$

where $a \gtrless \xi \lessgtr x$ or *$\xi = a + \theta(x - a)$, with $0 < \theta < 1$.*

For $a = 0$ we obtain **Maclaurin's formula**

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + R_n,$$

$$R_n = \frac{f^{(n+1)}(\theta x)}{(n + 1)!}x^{n+1}, \quad 0 < \theta < 1.$$

Here are the expansions of some functions by Maclaurin's formula:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + R_n; \quad R_n = \frac{e^{\theta x}}{(n + 1)!}x^{n+1};$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{x^{2m-1}(-1)^{m+1}}{(2m - 1)!} + R_{2m};$$

$$R_{2m} = (-1)^m \cos\theta x \cdot \frac{x^{2m+1}}{(2m + 1)};$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{x^{2m}(-1)^m}{(2m)!} + R_{2m+1};$$

$$R_{2m+1} = (-1)^{m+1}\cos\theta x \cdot \frac{x^{2m+2}}{(2m+2)!};$$

$$(1+x)^m = 1 + \frac{m}{1!}x + \frac{m(m-1)}{2!}x^2 +$$
$$+ \frac{m(m-1)(m-2)}{3!}x^3 + \ldots + \frac{m(m-1)\ldots[m-(n-1)]}{n!}x^n + R_n;$$
$$R_n = \frac{m(m-1)\ldots(m-n)}{(n+1)!}x^{n+1}(1+\theta x)^{m-n-1}$$

($0 < \theta < 1$ everywhere).

982. Is Rolle's theorem valid for the function $f(x) = x^2 - 6x + 100$ if $a = 1$, $b = 5$? At what value of ξ?

Solution. Since the function $f(x)$ is continuous and differentiable at all values of x and its values at the end points of the closed interval [1, 5] are equal, $f(1) = f(5) = 95$, Rolle's theorem holds on that closed interval. The value of ξ can be determined from the equation $f'(x) = 2x - 6 = 0$, i.e. $\xi = 3$.

983. Is Rolle's theorem valid for the function $f(x) = \sqrt[3]{8x - x^2}$ if $a = 0, b = 8$? At what value of ξ?

Solution. The function $f(x) = \sqrt[3]{8x - x^2}$ is continuous at all values of x and possesses a derivative $f'(x) = (8 - 2x)/(3\sqrt[3]{(8x - x^2)^2})$ for $x \neq 0, x \neq 8$, that is, is differentiable on the open interval (0, 8). In addition, $f(0) = f(8) = 0$. Thus, on the interval [0, 8] Rolle's theorem is valid; indeed, $f'(x) = 0$ for $x = \xi = 4$.

984. Given the function $f(x) = \sqrt[3]{(x - 8)^2}$. Assume $a = 0, b = 16$. Then $f(0) = f(16) = 4$. However, the derivative $f'(x) = 2/(3\sqrt[3]{x - 8})$ does not vanish at any point belonging to the interval (0, 16). Does it contradict Rolle's theorem?

Solution. No, it does not, since there is no derivative at the point $x = 8$ of the interval (0, 16) and the conditions of Rolle's theorem are violated.

985. Show that the derivative of the polynomial $f(x) = x^3 - x^2 - x + 1$ possesses a real root on the open interval $(-1, 1)$.

Solution. We find the roots of the given polynomial $x^3 - x^2 - x + 1 = 0$, or $(x - 1)^2(x + 1) = 0$, i.e. $x_1 = x_2 = 1$, $x_3 = -1$. Since $f(-1) = f(1) = 0$, it follows that in accordance with Rolle's theorem $f'(x)$ possesses a root on the interval $(-1, 1)$. Next we find the roots of the derivative: $f'(x) = 3x^2 - 2x - 1 = 0$, i.e. $x_1 = -1/3$, $x_2 = 1$. Thus, we see that between the roots -1 and 1 of the function there is a root of the derivative equal to $-1/3$.

986. On the arc AB of the curve $y = 2x - x^2$ find the point M at which the tangent is parallel to the chord AB if $A(1; 1)$ and $B(3; -3)$.

Solution. The function $y = 2x - x^2$ is continuous and differentiable for all values of x. In accordance with Lagrange's theorem, between the two values $a = 1$ and $b = 3$ there is a value $x = \xi$, satisfying the equality $y(b) - y(a) = (b - a)y'(\xi)$, where $y' = 2 - 2x$. Substituting the requisite values, we obtain

$$y(3) - y(1) = (3 - 1)y'(\xi);\ (2 \cdot 3 - 3^2) - (2 \cdot 1 - 1^2)$$
$$= (3 - 1) \cdot (2 - 2\xi);\ -4 = 4(1 - \xi).$$

Hence $\xi = 2$, $y(2) = 0$. Thus, the point M has the coordinates (2; 0).

987. On the arc AB of the curve specified by the parametric equations $x = t^2$, $y = t^3$ find the point M at which the tangent is parallel to the chord AB if the points A and B are associated with the values $t = 1$ and $t = 3$.

Solution. The slope of the chord AB is equal to $\dfrac{y(3) - y(1)}{x(3) - x(1)}$ and the slope of the tangent at the point M (for $t = \xi$) is equal to $\dfrac{y_t'(\xi)}{x_t'(\xi)}$, where $x_t' = 2t$, $y_t' = 3t^2$.

To determine ξ by Cauchy's theorem we obtain the equation

$$\frac{y(3) - y(1)}{x(3) - x(1)} = \frac{y_t'(\xi)}{x_t'(\xi)}, \quad \text{or} \quad \frac{27 - 1}{9 - 1} = \frac{3\xi^2}{2\xi}, \quad \text{or} \quad \frac{13}{4} = \frac{3}{2}\xi,$$

i. e. $\xi = 13/6$. The value of ξ we have obtained satisfies the inequality $1 < \xi < 3$.

Substituting the value $t = \xi$ into the parametric equation of the curve, we get $x = 169/36$, $y = 2197/216$. Thus, the desired point is $M(169/36;\ 2197/216)$.

988. Represent the function $f(x) = \sqrt[3]{x}$ as a fifth-degree polynomial with respect to the binomial $x - 1$.

Solution. Let us calculate the values of the function $f(x) = x^{1/3}$ and of its derivatives up to the fifth order inclusive at $a = 1$:

$$f(1) = 1,\ \ f'(x) = \frac{1}{3}x^{-2/3},\ \ f'(1) = \frac{1}{3};$$
$$f''(x) = -\frac{2}{9}x^{-5/3},\ \ f''(1) = -\frac{2}{9};$$
$$f'''(x) = \frac{10}{27}x^{-8/3},\ \ f'''(1) = \frac{10}{27};$$
$$f^{\text{IV}}(x) = -\frac{80}{81}x^{-11/3},\ \ f^{\text{IV}}(1) = -\frac{80}{81};$$
$$f^{\text{V}}(x) = \frac{880}{243}x^{-14/3},\ \ f^{\text{V}}(1) = \frac{880}{243}.$$

Consequently, we obtain by Taylor's formula

$$\sqrt[3]{x} = 1 + \frac{1}{3}(x - 1) - \frac{2}{9 \cdot 2!}(x - 1)^2 + \frac{10}{27 \cdot 3!}(x - 1)^3$$
$$- \frac{80}{81 \cdot 4!}(x - 1)^4 + \frac{880}{243 \cdot 5!}(x - 5)^5 + R_5,$$

where

$$R_5 = \frac{f^{\text{VI}}(\xi)}{6!}(x - 1)^6 = -\frac{12320}{729 \cdot 6!} \cdot \xi^{-17/3}(x - 1)^6, \quad 1 < \xi < x.$$

989. Represent the function $f(x) = a^x (a > 0)$ as a polynomial of the third degree with respect to x.

Solution. We have

$$\begin{aligned} f(x) &= a^x, & f(0) &= 1, \\ f'(x) &= a^x \ln a, & f'(0) &= \ln a, \\ f''(x) &= a^x \ln^2 a, & f''(0) &= \ln^2 a, \\ f'''(x) &= a^x \ln^3 a, & f'''(0) &= \ln^3 a, \\ f^{IV}(x) &= a^x \ln^4 a, & f^{IV}(\theta x) &= \ln^4 a \cdot a^{\theta x}. \end{aligned}$$

We obtain by Maclaurin's formula

$$a^x = 1 + x\ln a + \frac{x^2 \ln^2 a}{2!} + \frac{x^3 \ln^3 a}{3!} + R_3,$$

where $R_3 = \dfrac{x^4 \ln^4 a}{4!} a^{\theta x}$, $0 < \theta < 1$.

990. Calculate the approximate value $\sqrt[3]{29}$ with an accuracy to within 10^{-3}.

Solution. We represent the given root as follows: $\sqrt[3]{29} = \sqrt[3]{27 + 2} = 3\left(1 + \dfrac{2}{27}\right)^{1/3}$. Then we make use of the binomial expansion

$$(1 + x)^m = 1 + \frac{m}{1!}x + \frac{m(m-1)}{2!}x^2 + \ldots + \frac{m(m-1)\ldots[m-n+1]}{n\,!}x^n + R_n.$$

Hence follows an approximate equality

$$(1 + x)^m \cong 1 + \frac{m}{1!}x + \frac{m(m-1)}{2!}x^2 + \ldots + \frac{m(m-1)\ldots(m-n+1)}{n!}x^n,$$

whose error

$$R_n = \frac{m(m-1)\ldots(m-n)}{(n+1)!}x^{n+1}(1 + \theta x)^{m-n-1}$$

can be made arbitrarily small for $|x| < 1$ and for a sufficiently large n.

Putting $x = 2/27$ and $m = 1/3$, we get

$$\sqrt[3]{29} \cong 3\left(1 + \frac{2}{81} - \frac{2 \cdot 2}{81 \cdot 81} + \frac{2 \cdot 2 \cdot 2 \cdot 5}{81^3} - \frac{2^2 \cdot 5}{81^4} + \ldots + R_n\right).$$

Estimating the values of the consecutive errors in the calculation of $3|R_n|$, we get

$$3|R_1| < \frac{3 \cdot 2 \cdot 2}{81^2} < 0.002, \quad 3|R_2| < \frac{3 \cdot 2 \cdot 2 \cdot 2 \cdot 5}{81^3} < 0.0003.$$

Consequently, to make calculations with the desired degree of accuracy, it is sufficient to take three terms which preceded the remainder R_2, i.e. $\sqrt[3]{29} \cong 3(1 + 0.024 - 0.0006) = 3.072$.

991. Calculate $\sqrt{e}$ with an accuracy to within 0.0001.

Solution. Let us make use of Maclaurin's formula for the function e^x:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + R_n,$$

where $R_n = \dfrac{e^{\theta x}}{(n+1)!} x^{n+1}$, $0 < \theta < 1$. Putting $x = 1/2$, we get

$$\sqrt{e} = 1 + \frac{1}{2\cdot 1!} + \frac{1}{2^2\cdot 2!} + \frac{3}{2^3\cdot 3!} + \ldots + \frac{1}{2^n\cdot n!} + R_n,$$

where $R_n = \dfrac{e^{\theta/2}}{2^{n+1}(n+1)!}$, $0 < \theta < 1$.

Since $0 < \theta < 1$, $2 < e < 3$, it follows that $R_n < \dfrac{e^{1/2}}{2^{n+1}(n+1)!}$. But $e^{1/2} < 2$, therefore $R_n < \dfrac{1}{2^n(n+1)!}$. We must specify n so that the inequality $R_n < 0.0001$ be satisfied.

If $n = 3$, then $R_3 < \dfrac{1}{8\cdot 24}$; $R_3 < \dfrac{1}{192}$,

if $n = 4$, then $R_4 < \dfrac{1}{16\cdot 120}$; $R_4 < \dfrac{1}{1920}$,

if $n = 5$, then $R_5 < \dfrac{1}{32\cdot 720}$; $R_5 < 0.0001$.

To determine $\sqrt{e}$ with an accuracy to within 0.0001, we obtain an approximate equality

$$\sqrt{e} \cong 1 + \frac{1}{2} + \frac{1}{2^2\cdot 2!} + \frac{1}{2^3\cdot 3!} + \frac{1}{2^4\cdot 4!} + \frac{1}{2^5\cdot 5!}.$$

We perform summation, having turned all the summands into decimal fractions with one extra (reserve) decimal digit:

$$\begin{array}{r} 1.50000 \\ 0.12500 \\ 0.02083 \\ 0.00260 \\ 0.00026 \\ \hline 1.64869. \end{array}$$

Thus, we have $\sqrt{e} \cong 1.6487$.

992. Given the function $f(x)$ continuous together with its derivatives up to the $(n-1)$th order inclusive on the closed interval $[a, b]$ and possessing the nth-order derivative on the interval (a, b), with the equalities $f(a) = f(x_1) = f(x_2) = \ldots$ $\ldots = f(x_{n-1}) = f(b)$, where $a < x_1 < x_2 < \ldots < x_{n-1} < b$, holding true for that function. Prove that there is at least one point ξ on the interval (a, b) for which $f^{(n)}(\xi) = 0$.

993. Consider a special case of the preceding problem if $f(x) = (x-1) \times \times (x-2)(x-3)(x-4)$, $a = 1, x_1 = 2, x_2 = 3, b = 4$. Determine ξ.

994. Represent the function $1/x$ as a third-degree polynomial with respect to $x - x_0$ $(x_0 \neq 0)$.

995. At what point of the arc AB of the curve $y = x^3 - 3x$ is the tangent parallel to the chord AB if $A(0; 0)$, $B(3; 18)$?

Calculate with an accuracy to within 10^{-3}:

996. $\cos 41°$. **997.** $\sqrt[3]{121}$. **998.** $\sqrt[3]{e}$. **999.** $\sqrt[7]{129}$. **1000.** $\sin 36°$.

7.2.2. L'Hopital's rule for evaluating indeterminate forms. Assume that the functions $f(x)$ and $\varphi(x)$ are differentiable in the ε-neighbourhood of the point x_0 and $\varphi'(x) \neq 0$. If $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} \varphi(x) = 0$ or $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} \varphi(x) = \infty$, that is, at the point $x = x_0$ the quotient is the indeterminate form $0/0$ or ∞/∞, then

$$\lim_{x \to x_0} \frac{f(x)}{\varphi(x)} = \lim_{x \to x_0} \frac{f'(x)}{\varphi'(x)}$$

under the condition that there exists a limit of the ratio of the derivatives.

If at the point $x = x_0$ the quotient $f'(x)/\varphi'(x)$ is the indeterminate form $0/0$ or ∞/∞ and the derivatives $f'(x)$ and $\varphi'(x)$ comply with the respective conditions, we should turn to the ratio of the second derivatives and so on.

In the case of the indeterminate form $0 \cdot \infty$ or $\infty - \infty$, the given function should be transformed algebraically to reduce it to the indeterminate form $0/0$ or ∞/∞ and then L'Hopital's rule should be applied.

If we have the indeterminate form 0^0 or ∞^0 or 1^∞, we should take the logarithm of the given function and find the limit of its logarithm.

1001. Find $\lim\limits_{x \to 1} \dfrac{x^2 - 1 + \ln x}{e^x - e}$.

Solution. The numerator and the denominator tend to zero as $x \to 1$ and so we have the indeterminate form $0/0$. We use L'Hopital's rule, that is consider the limit of the ratio of the derivatives of the given functions:

$$\lim_{x \to 1} \frac{x^2 - 1 + \ln x}{e^x - e} = \lim_{x \to 1} \frac{2x + 1/x}{e^x} = \frac{3}{e}.$$

1002. Find $\lim\limits_{x \to 0} \dfrac{x - \sin x}{x^3}$.

Solution. This is the indeterminate form $0/0$. We have

$$\lim_{x \to 0} \frac{x - \sin x}{x^3} = \lim_{x \to 0} \frac{1 - \cos x}{3x^2} = \lim_{x \to 0} \frac{\sin x}{6x} = \frac{1}{6},$$

since $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$. Here L'Hopital's rule is applied twice.

1003. Find $\lim\limits_{x\to\infty} \dfrac{x^n}{e^x}$ if n is a positive integer.

Solution. This is the indeterminate form ∞/∞. We apply L'Hopital's rule n times:

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} = \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x} = \dots$$

$$\dots = \lim_{x\to\infty} \frac{n(n-1)(n-2)\dots 1}{e^x} = 0.$$

1004. Find $\lim\limits_{x\to\infty} \dfrac{xe^{x/2}}{x+e^x}$.

Solution. Here we also have the indeterminate form ∞/∞. We find

$$\lim_{x\to\infty} \frac{xe^{x/2}}{x+e^x} = \lim_{x\to\infty} \frac{e^{x/2}\left(1+\frac{x}{2}\right)}{1+e^x} = \lim_{x\to\infty} \frac{\frac{1}{2}e^{x/2}\left(2+\frac{x}{2}\right)}{e^x}$$

$$= \frac{1}{2} \lim_{x\to\infty} \frac{2+\frac{x}{2}}{e^{x/2}} = \frac{1}{2} \lim_{x\to\infty} \frac{1/2}{(1/2)e^{x/2}} = 0.$$

1005. Find $\lim\limits_{x\to 0} (x^2 \cdot \ln x)$.

Solution. Here is the indeterminate form $0 \cdot \infty$. We represent the product of the functions as a quotient and, having obtained an indeterminate form ∞/∞, apply L'Hopital's rule:

$$\lim_{x\to 0} (x^2 \cdot \ln x) = \lim_{x\to 0} \frac{\ln x}{1/x^2} = \lim_{x\to 0} \frac{1/x}{-2/x^3} = -\frac{1}{2} \lim_{x\to 0} x^2 = 0.$$

1006. Find $\lim\limits_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{e^x-1}\right)$.

Solution. This is the indeterminate form $\infty - \infty$. To find the limit of the function, we reduce the fractions to a common denominator, and then, having obtained an indeterminate form 0/0, apply L'Hopital's rule:

$$\lim_{x\to 0} \frac{e^x-1-x}{x(e^x-1)} = \lim_{x\to 0} \frac{e^x-1}{e^x-1+xe^x} = \lim_{x\to 0} \frac{e^x}{e^x(2+x)} = \frac{1}{2}.$$

1007. Find $\lim\limits_{x\to 0} (\sin x)^x$.

Solution. This is the indeterminate form 0^0. We designate the given function as y, i.e. $y = (\sin x)^x$, and take its logarithm:

$$\ln y = x \cdot \ln\sin x = \frac{\ln\sin x}{1/x}.$$

Next we calculate the limit of the logarithm of the given function applying L'Hopital's rule (here we have the indeterminate form ∞/∞):

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln\sin x}{1/x} = \lim_{x\to 0} \frac{\cos x/\sin x}{-1/x^2} = -\lim_{x\to 0} \frac{x^2\cos x}{\sin x}$$

$$= -\lim_{x\to 0}\left(x \cdot \cos x \cdot \frac{x}{\sin x}\right) = 0.$$

Consequently we have $\lim\limits_{x\to 0} y = e^0 = 1.$

1008. Find $\lim\limits_{x\to \pi/2} (\tan x)^{2\cos x}$.

Solution. This is the indeterminate form ∞^0. We put $(\tan x)^{2\cos x} = y$ and take the logarithm:

$$\ln y = 2\cos x \cdot \ln\tan x = \frac{2\ln\tan x}{1/\cos x}.$$

Applying L'Hopital's rule, we obtain

$$\lim_{x\to\pi/2} \ln y = 2\lim_{x\to\pi/2} \frac{\ln\tan x}{\sec x} = 2\lim_{x\to\pi/2} \frac{\sec^2 x/\tan x}{\sec x \cdot \tan x}$$

$$= 2\lim_{x\to\pi/2} \frac{\sec\ x}{\tan^2 x} = 2\lim_{x\to\pi/2} \frac{\sec x \cdot \tan x}{2\tan x \cdot \sec^2 x} = \lim_{x\to\pi/2} \cos x = 0,$$

i.e. $\lim\limits_{x\to\pi/2} y = e^0 = 1.$

1009. Find $\lim\limits_{x\to 0} (1 + x)^{\ln x}$.

Solution. This is the indeterminate form 1^∞. Taking the logarithm and applying L'Hopital's rule, we get

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \ln x \cdot \ln(1 + x) = \lim_{x\to 0} \frac{\ln(1 + x)}{1/\ln x}$$

$$= \lim_{x\to 0} \frac{1/(1 + x)}{-1/(x\ln^2 x)} = -\lim_{x\to 0} \frac{x\ln^2 x}{x+1} = -\lim_{x\to 0} \frac{\ln^2 x}{1+1/x}$$

$$= -\lim_{x\to 0} \frac{(2\ln x)/x}{-1/x^2} = 2\lim_{x\to 0} \frac{\ln x}{1/x} = 2\lim_{x\to 0} \frac{1/x}{-1/x^2} = 0.$$

Thus, $\lim\limits_{x\to 0} y = e^0 = 1.$

Find the limits of the following functions:

Indeterminate form 0/0.

1010. $\lim\limits_{x\to 1} \dfrac{x^3 - 3x^2 + 2}{x^3 - 4x^2 + 3}$. **1011.** $\lim\limits_{x\to 0} \dfrac{e^x - e^{-x}}{\ln(1 + x)}$.

1012. $\lim\limits_{x\to\infty} \dfrac{\pi - 2\arctan x}{e^{3/x} - 1}$. **1013.** $\lim\limits_{x\to 0} \dfrac{2 - (e^x + e^{-x})\cos x}{x^4}$.

1014. $\lim\limits_{x\to 0} \dfrac{e^{3x} - 3x - 1}{\sin^2 5x}$. **1015.** $\lim\limits_{x\to 0} \dfrac{\sin 3x - 3xe^x + 3x^2}{\arctan x - \sin x - x^3/6}$.

Indeterminate form ∞/∞.

1016. $\lim\limits_{x\to a} \dfrac{\ln(x-a)}{\ln(e^x - e^a)}$. **1017.** $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^n}$ $(n > 0)$.

1018. $\lim\limits_{x\to 0} \dfrac{\ln x}{1 + 2\ln\sin x}$. **1019.** $\lim\limits_{x\to 1} \dfrac{\tan(\pi x/2)}{\ln(1-x)}$. **1020.** $\lim\limits_{x\to 1} \dfrac{\ln(x-1)}{\cot \pi x}$.

Indeterminate form $0 \cdot \infty$.

1021. $\lim\limits_{x\to 0} (x \cdot \cot \pi x)$. **1022.** $\lim\limits_{x\to 0} (\arcsin x \cdot \cot x)$.

1023. $\lim\limits_{x\to 0} (1 - \cos x) \cdot \cot x$.

Indeterminate form $\infty - \infty$.

1024. $\lim\limits_{x\to 1} \left(\dfrac{1}{x-1} - \dfrac{1}{\ln x}\right)$. **1025.** $\lim\limits_{x\to 1} \left(\dfrac{p}{1-x^p} - \dfrac{q}{1-x^q}\right)$.

1026. $\lim\limits_{x\to 0} \left(\dfrac{1}{x^2} - \cot^2 x\right)$.

Indeterminate forms 0^0, ∞^0, 1^∞.

1027. $\lim\limits_{x\to \pi/2} (\pi - 2x)^{\cos x}$. **1028.** $\lim\limits_{x\to 0} (\cos 2x)^{3/x^2}$.

1029. $\lim\limits_{x\to\infty} (x + 2^x)^{1/x}$. **1030.** $\lim\limits_{x\to 0} \left(\dfrac{\tan x}{x}\right)^{1/x^2}$.

7.2.3. Increasing and decreasing functions. Extremum of a function. The function $f(x)$ is said to be *increasing at a point* x_0 if for a sufficiently small $h > 0$ the condition (Fig. 29)

$$f(x_0 - h) < f(x_0) < f(x_0 + h)$$

is fulfilled.

The function $f(x)$ is said to be *decreasing at a point* x_0 if for a sufficiently small $h > 0$ the condition (Fig. 30)

$$f(x_0 - h) > f(x_0) > f(x_0 + h)$$

is fulfilled.

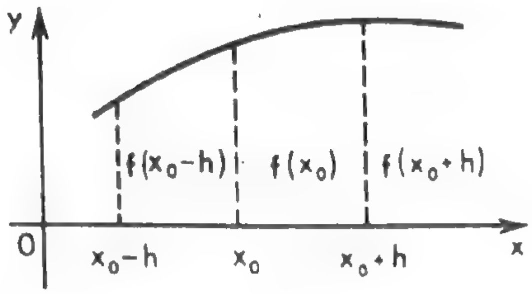

Fig. 29

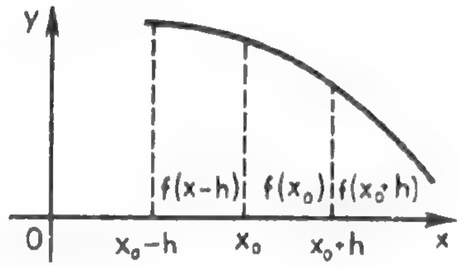

Fig. 30

The function $f(x)$ is called *increasing on the open interval* (a, b) if for any two points x_1 and x_2 belonging to the indicated interval and satisfying the inequality $x_1 < x_2$ the inequality $f(x_1) < f(x_2)$ holds true.

The function $f(x)$ is called *decreasing on the open interval* (a, b) if for any points x_1 and x_2 belonging to the indicated interval and satisfying the inequality $x_1 < x_2$ the inequality $f(x_1) > f(x_2)$ holds true.

Tests for increase and decrease of a function.

(1) *If* $f'(x_0) > 0$, *then the function* $f(x)$ *increases at the point* x_0.

(2) *If* $f'(x_0) < 0$, *then the function* $f(x)$ *decreases at the point* x_0.

The value $f(x_0)$ is said to be the *maximum* of the function $f(x)$ if for a sufficiently small $h > 0$ the condition

$$f(x_0 - h) < f(x_0) \quad \text{and} \quad f(x_0 + h) < f(x_0)$$

is complied with. In this case the point x_0 is called *the point of maximum* of the function $f(x)$ (Fig. 31).

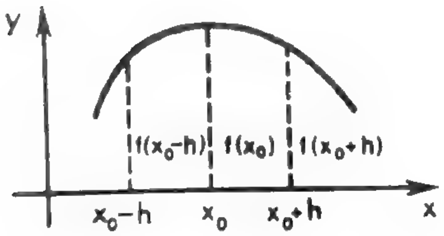

Fig. 31

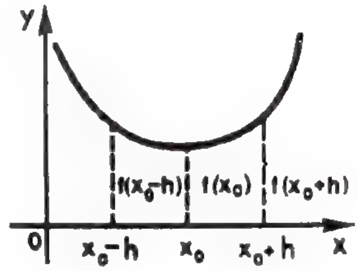

Fig. 32

The value $f(x_0)$ is said to be the *minimum* of the function $f(x)$ if for a sufficiently small $h > 0$ the condition

$$f(x_0 - h) > f(x_0) \quad \text{and} \quad f(x_0 + h) > f(x_0)$$

is complied with. In this case the point x_0 is called *the point of minimum* of the function $f(x)$ (Fig. 32).

The maximum and the minimum of a function are called the *extrema* of the function. The point of maximum or minimum of a function is called the *point of its extremum*.

A necessary condition for an extremum. *If the function $f(x)$ possesses an extremum at a point x_0, then the derivative $f'(x_0)$ vanishes or does not exist.*

The point x_0 at which $f'(x_0) = 0$ is called a *stationary point*. The points at which $f'(x) = 0$ or $f'(x)$ does not exist are called *critical points*. Not every critical point is a point of extremum.

Sufficient conditions for an extremum.

Rule 1. If x_0 is a critical point of the function $f(x)$ and the inequalities $f'(x_0 - h) > 0, f'(x_0 + h) < 0$ are satisfied for an arbitrary, sufficiently small $h > 0$, then the function $f(x)$ possesses a maximum at the point x_0; now if $f'(x_0 - h) < 0, f'(x_0 + h) > 0$, then the function $f(x)$ possesses a minimum at the point x_0.

If the signs of $f'(x_0 - h)$ and $f'(x_0 + h)$ are the same, then the function $f(x)$ does not possess an extremum at the point x_0.

Rule 2. *If $f'(x_0) = 0, f''(x_0) \neq 0$, then the function $f(x)$ possesses an extremum at the point x_0, namely, a maximum if $f''(x_0) < 0$ and a minimum if $f''(x_0) > 0$.*

Rule 3. *Assume $f'(x_0) = 0, f''(x_0) = 0, \ldots, f^{(n-1)}(x_0) = 0, f^{(n)}(x_0) \neq 0$. In that case, the function $f(x)$ possesses an extremum at the point x_0 if n is an even number, namely, a maximum for $f^{(n)}(x_0) < 0$ and a minimum for $f^{(n)}(x_0) > 0$. Now if n is an odd number, then the function $f(x)$ does not possess an extremum at the point x_0.*

To find the greatest (the least) value of the function $f(x)$ on the closed interval $[a, b]$ it is necessary to choose the greatest (the least) value out of the values of the function at the end points of the interval and at the critical points belonging to that interval.

1031. Given the points $x = 3$, $x = 1$, $x = -1$, $x = 0.5$. At which of the enumerated points does the function $y = x^3 - 3x^2$ increase? Decrease?

Solution. Let us find the derivative $y' = 3x^2 - 6x$. We have:

if $x = 3$, then $y' = 9 > 0$ and the function increases;
if $x = 1$, then $y' = -3 < 0$ and the function decreases;
if $x = -1$, then $y' = 9 > 0$ and the function increases;
if $x = 0.5$, then $y' = -2.25 < 0$ and the function decreases.

1032. Find the intervals of increase and decrease of the function $y = x(1 + \sqrt{x})$.

Solution. We find $y' = 1 + (3/2)x^{1/2}$. Since the derivative is positive on the interval $[0, +\infty]$, the function increases throughout the domain of its definition.

1033. Find the intervals of increase and decrease of the function $y = x - 2\sin x$ if $0 \leqslant x \leqslant 2\pi$.

Solution. Let us find the derivative: $y' = 1 - 2\cos x$. It is evident that $y' > 0$ on the interval $(\pi/3, 5\pi/3)$ and $y' < 0$ on the intervals $(0, \pi/3)$ and $(5\pi/3, 2\pi)$. Thus, on the interval $(\pi/3, 5\pi/3)$ the given function increases and on the intervals $(0, \pi/3)$ and $(5\pi/3, 2\pi)$ it decreases.

1034. Test the function $y = (x - 5)e^x$ for an extremum.

Solution. We find the derivative $y' = (x - 4)e^x$, equate it to zero and find the

stationary point: $e^x(x-4)=0$, $x=4$; $y'(4-h)=-he^{4-h}<0$, $y'(4+h)=he^{4+h}>0$.

In accordance with Rule 1 we infer that the function possesses a minimum $y_{min}=-e^4$ at the point $x=4$.

1035. Test the function $y=x\sqrt{1-x^2}$ for an extremum.

Solution. The function is defined at $-1\leqslant x\leqslant 1$. We find the derivative: $y'=(1-2x^2)/\sqrt{1-x^2}$; $y'=0$ at $1-2x^2=0$; hence $x_1=-1\sqrt{2}$, $x_2=1/\sqrt{2}$ (stationary points); $y'=\infty$ at $x=\pm 1$, that is, on the boundaries of the domain of definition of the function.

Next we find the second derivative: $y''=x(2x^2-3)/(1-x^2)^{3/2}$. We calculate the values of the second derivative at the stationary points. At $x=1/\sqrt{2}$ we have

$$y''(1/\sqrt{2})=\frac{1\cdot(1-3)}{\sqrt{2}(1-1/2)^{3/2}}<0,$$

consequently, in accordance with Rule 2 we infer that the function possesses a maximum $y_{max}=(1/\sqrt{2})\sqrt{1/\sqrt{2}}=1/2$ at the point $x=1/\sqrt{2}$. At $x=-1/\sqrt{2}$ we get

$$y''(-1/\sqrt{2})=-\frac{1\cdot(1-3)}{\sqrt{2}(1-1/2)^{3/2}}>0,$$

that is, at the point $x=-1/\sqrt{2}$ the function possesses a minimum $y_{min}=-1/2$.

There is no extremum at the critical points $x=\pm 1$ because in accordance with the definition only the interior points of the domain of definition of a function can serve as the points of extremum.

1036. Test the function $y=(x-1)^4$ for a possible extremum.

Solution. We find the derivative: $y'=4(x-1)^3$; $(x-1)^3=0$; $x=1$ is a stationary point. The second derivative $y''=12(x-1)^2$ is equal to zero at $x=1$. The third derivative $y'''=24(x-1)$ also vanishes at $x=1$. The fourth derivative $y^{IV}=24>0$. Therefore, in accordance with Rule 3, we infer that at the point $x=1$ the function possesses a minimum $y_{min}=0$.

1037. Test the function $y=1-(x-2)^{4/5}$ for a possible extremum.

Solution. We find $y'=-\frac{4}{5}(x-2)^{-1/5}=-\frac{4}{5\sqrt[5]{x-2}}$. The derivative does not vanish at any values of x and does not exist only at $x=2$ (critical point).

Since the inequalities $y'(2-h)>0$ and $y'(2+h)<0$ hold true for a sufficiently small positive h, we infer, in accordance with Rule 1, that at $x=2$ the function possesses a maximum $y_{max}=1$.

1038. Test the function $y=(x-2)^{2/3}(2x+1)$ for a possible extremum.

Solution. We find $y'=\frac{10}{3}\cdot\frac{x-1}{\sqrt[3]{x-2}}$. The critical points are $x=1$ (the deriva-

tive is zero) and $x = 2$ (the derivative does not exist). The inequalities $y'(1 - h) > 0$, $y'(1 + h) < 0$, $y'(2 - h) < 0$, $y'(2 + h) > 0$ hold at a sufficiently small $h > 0$. Consequently, at the point $x = 1$ the function possesses a maximum $y_{max} = 3$ and at the point $x = 2$ it possesses a minimum $y_{min} = 0$.

1039. Find the greatest and the least value of the function $f(x) = 3x - x^3$ on the closed interval $[-2, 3]$.

Solution. We find the derivative: $f'(x) = 3 - 3x^2$; $3 - 3x^2 = 0$, i.e. $x = \pm 1$ are stationary points. Next we determine the values of the function at these points: $f(1) = 2$, $f(-1) = -2$. Then we calculate the values of the given function at the end points of the closed interval: $f(-2) = 2$, $f(3) = -18$. From the four values we have obtained, we choose the greatest and the least value.

Thus, the greatest value of the function on the given interval is equal to 2 and the least, to -18.

1040. Find a cylinder which would have the greatest volume for a given total area S.

Solution. Suppose the radius of the base of the cylinder is x and the altitude is y. Then we have

$$S = 2\pi x^2 + 2\pi xy, \quad \text{i.e.} \quad y = \frac{S - 2\pi x^2}{2\pi x} = \frac{1}{2\pi}\left(\frac{S}{x} - 2\pi x\right).$$

Consequently, the volume of the cylinder will be expressed as

$$V = V(x) = \pi x^2 \frac{1}{2\pi}\left(\frac{S}{x} - 2\pi x\right) = \frac{S}{2}x - \pi x^3.$$

The problem reduces to testing the function $V(x)$ for a possible maximum at $x > 0$.

We find the derivative $\frac{dV}{dx} = \frac{S}{2} - 3\pi x^2$ and equate it to zero, whence we get $x = \sqrt{S/(6\pi)}$.

Then we find the second derivative: $\frac{d^2V}{dx^2} = -6\pi x$. Since the condition $\frac{d^2V}{dx^2} < 0$ is complied with at $x = \sqrt{S/(6\pi)}$, the volume has the greatest value, with

$$y = \frac{S - 2\pi \cdot S/(6\pi)}{2\pi\sqrt{S/(6\pi)}} = 2\sqrt{S/(6\pi)} = 2x,$$

that is, the axial section of the cylinder must be a square.

Find the intervals of increase and decrease of the following functions:

1041. $y = 2 - 3x + x^3$. **1042.** $y = (x^2 - 1)^{3/2}$.
1043. $y = xe^{-x}$. **1044.** $y = (2 - x)(x + 1)^2$.

Find the extrema of the following functions:

1045. $y = x^2(1 - x\sqrt{x})$ **1046.** $y = x + \sqrt{3 - x}$.
1047. $y = \ln(x^2 + 1)$. **1048.** $y = \cosh^2 x$. **1049.** $y = \dfrac{x}{\ln x}$.
1050. $y = xe^{-x^2/2}$. **1051.** $y = (x - 1)^{6/7}$.
1052. $y = (2x - 1)\sqrt[3]{(x - 3)^2}$. **1053.** $y = x^4 - 4x^3 + 6x^2 - 4x$.
1054. $y = x - 2\sin^2 x$. **1055.** $y = e^{1.5 \sin x}$.

1056. Find the least and the greatest value of the function $y = x^4 - 2x^2 + 3$ on the closed interval $[-3, 2]$.

1057. Find the point on the Oy axis from which the closed interval $[AB]$ is seen at the greatest angle if $A(2; 0)$, $B(8; 0)$.

1058. Point B is 60 km from the railway. The distance from point A to point C, nearest to point B, is 285 km by railway. At what distance from point C should a railway station be built in order that the time required for travel between points A and B be minimal if the speed by rail is 52 km/h and by the highroad, 20 km/h?

1059. Compute the sides of rectangle of the greatest area which can be inscribed into the ellipse $x^2/25 + y^2/9 = 1$.

1060. The wire of length l has been bent to form a rectangle. What are the dimensions of the rectangle if its area is the greatest?

1061. Find the greatest volume of the cone whose generating element is equal to l.

1062. Find the greatest volume of the cylinder whose total surface area is equal to S.

1063. A tourist is going from point A, located at the highroad, to point B which is 8 km from the highroad. The distance from A to B along a straight line is 17 km. At what point should the tourist turn off the highroad to reach point B in the shortest time if his speed by the highroad is 5 km/h and after he has left the highroad, 3 km/h?

1064. A channel 27 m wide falls at a right angle into another channel 64 m wide. What is the greatest length of the logs which can be floated along this system of channels?

1065. At what height above the centre of a round table of radius a should an electric bulb be hung to attain the greatest illumination at the edge of the table?

Hint. Lighting brilliancy is expressed by the formula $I = (k \cdot \sin\varphi)/r^2$, where φ is the angle of incidence of the rays, r is the distance between the source of light and the illuminated area, k is the intensity of the source of light.

7.2.4. Convexity. Concavity. Point of Inflection. The graph of the function $y = f(x)$ is said to be *convex* on the interval (a, b) if it lies below the tangent drawn at any point of that interval (Fig. 33).

The graph of the function $y = f(x)$ is said to be *concave* on the interval (a, b) if it lies above the tangent drawn at any point of that interval (Fig. 34).

The sufficient condition for the convexity (concavity) of the graph of a function. *If $f''(x) < 0$ on the open interval (a, b) then the graph of the function is convex on that interval; now if $f''(x) > 0$, then on the open interval (a, b) the graph of the function is concave.*

Fig. 33

Fig. 34

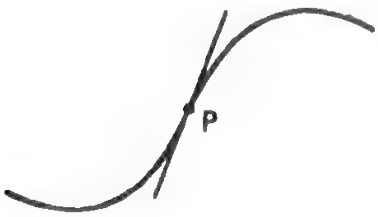

Fig. 35

The point $(x_0; f(x_0))$ of the graph of the function separating its convex part from the concave part is called a *point of inflection* (Fig. 35).

If x_0 is the abscissa of the inflection point of the graph of the function $y = f(x)$, then the second derivative is equal to zero or does not exist. The points at which $f''(x) = 0$ or $f''(x)$ does not exist are called *critical points of the 2nd kind.*

If x_0 is a critical point of the 2nd kind and the inequalities $f''(x_0 - h) < 0$, $f''(x_0 + h) > 0$ (or the inequalities $f''(x_0 - h) > 0$, $f''(x_0 + h) < 0$) hold for an arbitrary sufficiently small $h > 0$, then the point of the curve $y = f(x)$ with the abscissa x_0 is a point of inflection.

If $f''(x_0 - h)$ and $f''(x_0 + h)$ are of the same sign, then the point of the curve $y = f(x)$ with the abscissa x_0 is not a point of inflection.

1066. Find the intervals of convexity and concavity of the graph of the function $y = x^5 + 5x - 6$.

Solution. We have $y' = 5x^4 + 5$, $y'' = 20x^3$. If $x < 0$, then $y'' < 0$ and the curve is convex; now if $x > 0$, then $y'' > 0$ and the curve is concave. Thus we see that the curve is convex on the interval $(-\infty, 0)$ and concave on the interval $(0, +\infty)$.

1067. Find the extrema of the function $y = (x + 1)^2(x - 2)$ and the points of inflection of its graph.

Solution. Let us find the first derivative: $y' = 3(x^2 - 1)$. The roots of the first derivative $x_1 = -1$, $x_2 = 1$. Let us now find the second derivative: $y'' = 6x$. Then we find the values of the second derivative at the stationary points: $y''(-1) = -6 < 0$, i.e. $y_{max} = 0$; $y''(1) = 6 > 0$, i.e. $y_{min} = -4$.

We shall now find the inflection point, for which purpose we shall equate the second derivative to zero: $6x = 0$, i.e. $x = 0$. To the left of the point $x = 0$ we have $y''(0 - h) < 0$, the curve is convex, and to the right of the point $x = 0$ we have $y''(0 + h) > 0$, the curve is concave; consequently, the point with the abscissa $x = 0$ is an inflection point; $y_{inf.p.} = -2$.

1068. Find the inflection points of the curve $y = (x - 5)^{5/3} + 2$.

Solution. We find

$$y' = \frac{5}{3}(x - 5)^{2/3}, \quad y'' = \frac{10}{9\sqrt[3]{x - 5)}}.$$

The second derivative does not vanish for any value of x and does not exist at $x = 5$. The value $x = 5$ is the abscissa of the inflection point since $y''(5 - h) < 0$, $y''(5 + h) > 0$. Thus, (5; 2) is the inflection point.

1069. Find the intervals of convexity and concavity of the curve $y = xe^x$.

1070. Find the inflection points of the curve $y = (x - 4)^5 + 4x + 4$.

1071. Find the inflection points of the curve $y = (x - 1\sqrt[7]{(x - 1)^6}$.

1072. Find the inflection points of the curve $y = x^4 - 8x^3 + 24x^2$.

7.2.5. Asymptotes. The straight line L is an *asymptote* to the curve $y = f(x)$ if the distance from the point $M(x; y)$ of the curve to the line L tends to zero as this point recedes indefinitely from the origin along the curve (that is, as at least one of the coordinates of the point tends to infinity).

The straight line $x = a$ is a *vertical asymptote* of the curve $y = f(x)$ if $\lim\limits_{x \to a} f(x) = +\infty$ or $\lim\limits_{x \to a} f(x) = -\infty$.

The straight line $y = b$ is a *horizontal asymptote* of the curve $y = f(x)$ if there is a limit $\lim\limits_{x \to +\infty} f(x) = b$ or $\lim\limits_{x \to -\infty} f(x) = b$.

The straight line $y = kx + b$ is an *inclined asymptote* of the curve $y = f(x)$ if there are limits

$$k = \lim_{x \to +\infty} \frac{f(x)}{x}, \quad b = \lim_{x \to +\infty} [f(x) - kx]$$

or

$$k = \lim_{x \to -\infty} \frac{f(x)}{x}, \quad b = \lim_{x \to -\infty} [f(x) - kx].$$

1073. Find the asymptotes of the curve $y = \sqrt{x^3/(x - 2)}$.

Solution. The function is defined on the intervals $(-\infty, 0)$ and $(2, +\infty)$. Since $\lim\limits_{x \to 2 + 0} \sqrt{x^3/(x - 2)}$ the line $x = 2$ is a vertical asymptote of the curve.

The curve does not have horizontal asymptotes since $\lim\limits_{x \to +\infty} \sqrt{x^3/(x - 2)}$ and $\lim\limits_{x \to -\infty} \sqrt{x^3(x - 2)}$ are not finite values.

Let us determine whether there are inclined asymptotes.

$$(1)\ k_1 = \lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{\sqrt{x^3/(x - 2)}}{x} = \lim_{x \to +\infty} \sqrt{\frac{x}{x - 2}} =$$

$$= \lim_{x \to +\infty} \sqrt{\frac{1}{1 - 2/x}} = 1,$$

$$b_1 = \lim_{x\to+\infty} [f(x) - k_1x] = \lim_{x\to+\infty} \left(\sqrt{\frac{x^3}{x-2}} - x\right)$$

$$= \lim_{x\to+\infty} \frac{x(\sqrt{x} - \sqrt{x-2})}{\sqrt{x-2}} = \lim_{x\to+\infty} \frac{x(x - x + 2)}{\sqrt{x-2}(\sqrt{x} + \sqrt{x-2})}$$

$$= \lim_{x\to+\infty} \frac{2}{\sqrt{1-\frac{2}{x}}\left(1 + \sqrt{1 - \frac{2}{x}}\right)} = 1.$$

Thus we see that there is a right inclined asymptote $y = x + 1$;

$$(2)\ k_2 = \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{\sqrt{x^3/(x-2)}}{x} = \lim_{x\to-\infty} \frac{\sqrt{x/(x-2)}}{-1}$$

(we have divided the numerator and the denominator by the positive value $-x$), i.e.

$$k_2 = -\lim_{x\to-\infty} \sqrt{\frac{1}{1-2/x}} = -1,$$

$$b_2 = \lim_{x\to-\infty} [f(x) - k_2x] = \lim_{x\to-\infty} \left(\sqrt{\frac{x^3}{x-2}} + x\right)$$

$$= \lim_{x\to-\infty} \left(\sqrt{\frac{(-x)^3}{2-x}} + x\right) = \lim_{x\to-\infty} \frac{-x\sqrt{-x} + x\sqrt{2-x}}{\sqrt{2-x}}$$

$$= -\lim_{x\to-\infty} \frac{x(\sqrt{-x} - \sqrt{2-x})}{\sqrt{2-x}} = -\lim_{x\to-\infty} \frac{x(-x - 2 + x)}{\sqrt{2-x}(\sqrt{\;x} + \sqrt{2-x})} = -1.$$

Thus, there exists a left inclined asymptote $y = -x - 1$ (Fig. 36).

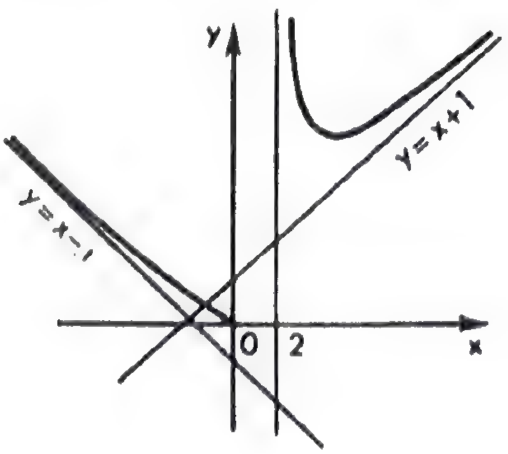

Fig. 36

1074. Find the asymptotes of the curve $y = x + 2 \arctan x$.

Solution. It is easy to see that the curve does not possess either vertical or horizontal asymptotes. Let us find inclined asymptotes:

$$(1)\ k_1 = \lim_{x \to +\infty} \frac{x + 2 \arctan x}{x} = \lim_{x \to +\infty} \left(1 + \frac{2 \arctan x}{x}\right) = 1.$$

$$b_1 = \lim_{x \to +\infty} (x + 2 \arctan x - x) = 2(\pi/2) = \pi;$$

$y = x + \pi$ is the right inclined asymptote;

$$(2)\ k_2 = \lim_{x \to -\infty} \frac{x + 2 \arctan x}{x} = \lim_{x \to -\infty} \left(1 + \frac{2 \arctan x}{x}\right) = 1.$$

$$b_2 = \lim_{x \to -\infty} (x + 2 \arctan x - x) = 2(-\pi/2) = -\pi;$$

$y = x - \pi$ is the left inclined asymptote.

1075. Find the asymptotes of the curve $y = x^2 e^{-x}$.

Solution. It is evident that there are no vertical asymptotes. If $x \to \infty$ then $y \to 0$. Consequently, the y-axis is a horizontal asymptote of the given curve. Let us determine whether there exists an inclined asymptote:

$$k = \lim_{x \to \infty} \frac{x^2 e^{-x}}{x} = \lim_{x \to \infty} \frac{x}{e^x} = 0.$$

Thus we see that there is only a horizontal asymptote, $y = 0$.

1076. Find the asymptote of the curve $y = \dfrac{x^2 - 2x + 3}{x + 2}$.

Solution. If $x \to -2$, then $y \to \infty$, i.e. $x = -2$ is a vertical asymptote. Let us find the nonvertical asymptotes:

$$k = \lim_{x \to \infty} \frac{x^2 - 2x + 3}{x(x + 2)} = 1, \quad b = \lim_{x \to \infty} \left[\frac{x^2 - 2x + 3}{x + 2} - x\right] = -4.$$

Thus, the inclined asymptote is specified by the equation $y = x - 4$.

Find the asymptotes of the curves:

1077. $y = 2x - \dfrac{\cos x}{x}$. **1078.** $y = \dfrac{\ln^2 x}{x} - 3x$. **1079.** $y = \sqrt{x^3 - 6x^2}$.

1080. $y = 0.5x + \arctan x$. **1081.** $y = -x \arctan x$.

7.2.6. Using characteristic points to construct the graphs of functions. When constructing the graph of the function $y = f(x)$, it is of use to elucidate its peculiarities. For that purpose it is necessary

(1) to find the domain of definition of the function;

(2) to test the function for evenness and oddness;

(3) to find the points of intersection of the graph of the function and the coordinate axes;

(4) to test the function for continuity; to find the points of discontinuity (if there are any) and establish the nature of the discontinuity; to find the asymptotes of the curve $y = f(x)$;

(5) to find the intervals of increase and decrease of the function and its extrema;

(6) to find the intervals of convexity and concavity of the curve and its points of inflection.

1082. Construct the graph of the function $y = \dfrac{x^3 + 4}{x^2}$.

Solution. (1) The domain of definition of the function is the entire x-axis except for the point $x = 0$, i.e. $D(y) = (-\infty, 0[\cup]0, +\infty)$.

(2) The function is neither even nor odd.

(3) Next we find the points of intersection of the graph and the x-axis; we have

$\dfrac{x^3 + 4}{x^2} = 0$; $x = -\sqrt[3]{4}$.

(4) The point of discontinuity $x = 0$, with $\lim\limits_{x \to 0} y = \infty$; consequently, $x = 0$ (the y-axis) is a vertical asymptote of the graph.

Let us now find the inclined asymptotes:

$$k = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{x^3 + 4}{x^3} = 1;$$

$$b = \lim_{x \to \infty} [f(x) - kx] = \lim_{x \to \infty} \left(\frac{x^3 + 4}{x^2} - x \right) = \lim_{x \to \infty} \frac{4}{x^2} = 0.$$

The inclined asymptote is specified by the equation $y = x$.

(5) Let us find the extrema of the function and the intervals of increase and decrease. We have $y' = 1 - 8/x^3 = (x^3 - 8)/x^3$; $y' = 0$ for $x = 2$; $y' = \infty$ for $x = 0$ (the point of discontinuity of the function). The points $x = 0$ and $x = 2$ divide the number axis into the intervals $(-\infty, 0)$, $(0, 2)$ and $(2, +\infty)$, with $y' > 0$ on the intervals $(-\infty, 0)$ and $(2, +\infty)$ (the function increases) and $y' < 0$ on the interval $(0, 2)$ (the function decreases).

Next we find $y'' = 24/x^4$; $y''(2) > 0$. Consequently, $x = 2$ is the point of minimum; $y_{min} = 3$.

(6) Then we find the intervals of convexity and concavity of the curve and its points of inflection. Since $y'' > 0$, the graph of the function is concave everywhere. The curve has no points of inflection.

Using the data obtained, we construct the graph of the function (Fig. 37).

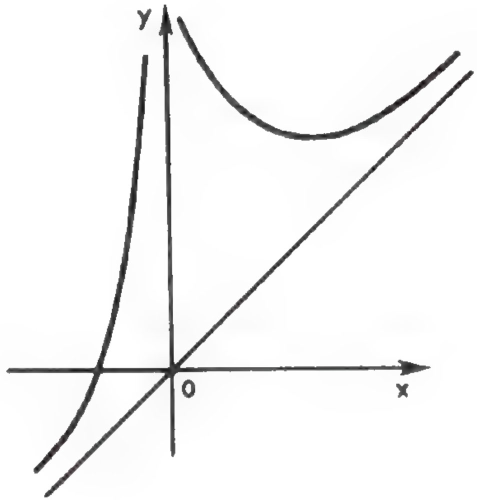

Fig. 37

1083. Construct the graph of the function $y = \sqrt[3]{1 - x^3}$.

Solution. (1) The domain of definition is the entire x-axis, i.e. $D(y) = (-\infty, +\infty)$.

(2) The function is neither even nor odd.

(3) The points of intersection with the coordinate axes: if $x = 0$, then $y = 1$; if $y = 0$, then $x = 1$.

(4) There are no points of discontinuity of vertical asymptotes. We have

$$k = \lim_{x \to \infty} \frac{\sqrt[3]{1 - x^3}}{x} = -1;$$

$$b = \lim_{x \to \infty} (\sqrt[3]{1 - x^3} + x) = \lim_{x \to \infty} \frac{1}{\sqrt[3]{(1 - x^3)^2} - x\sqrt[3]{1 - x^3} + x^2} = 0.$$

Thus, the inclined asymptote $y = -x$.

(5) We find $y' = -x^2/\sqrt[3]{(1 - x^3)^2}$; $y' = 0$ for $x = 0$; $y' = \infty$ for $x = 1$. The derivative does not change sign in the neighbourhood of the critical points, there are no extrema. Since $y' < 0$ for all $x \neq 0$, the function decreases throughout the number axis.

(6) We find $y'' = -2x/\sqrt[3]{(1 - x^3)^5}$; $y'' = 0$ for $x = 0$; $y'' = \infty$ for $x = 1$; $y''(-h) > 0$; $y''(h) < 0$, $y''(1 - h) < 0$; $y''(1 + h) > 0$. It follows that on the intervals $(-\infty, 0)$ and $(1, +\infty)$ the curve is concave and on the interval $(0, 1)$ it is convex. The points of inflection have the coordinates $(0; 1)$ and $(1; 0)$.

Using the data obtained, we construct the graph (Fig. 38).

Construct the graphs of the functions:

1084. $y = \sin^2 x$. **1085.** $y = 3\sqrt[3]{x} - x$.
1086. $y = \ln x - \ln (x - 1)$. **1087.** $y = \ln \dfrac{x}{x - 1}$.

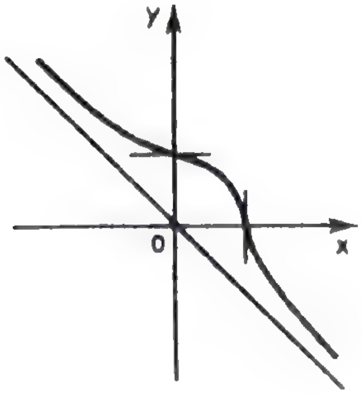

Fig. 38

1088. $y = \dfrac{x^3}{x^2 - 4}$. **1089.** $y = \dfrac{\ln x}{\sqrt{x}}$. **1090.** $y = 16x(x - 1)^3$.

1091. $y = (x - 1)\sqrt{x}$. **1092.** $y = x + e^{-x}$.
1093. $y = \ln(x + \sqrt{x^2 + 1})$. **1094.** $y = e^{2x - x^2}$.

1095. $y = \dfrac{x^3}{(x - 2)^2}$.

7.3. The Curvature of a Plane Curve

The *angle of contingence* of the arc AB of a plane curve is the angle φ between the tangent lines drawn at the points A and B of that curve (Fig. 39). The ratio be-

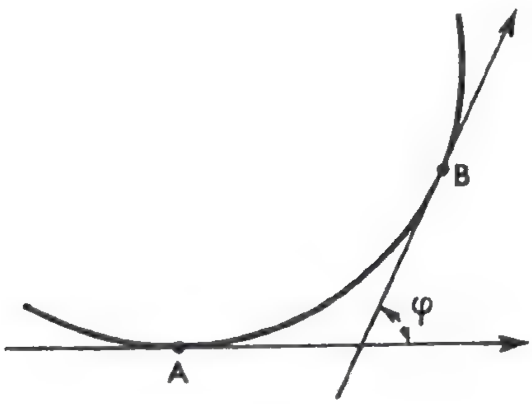

Fig. 39

tween the angle of contingence and the length s of the arc AB is called the *average curvature* along the arc AB:

$$k_{av} = \varphi/s.$$

The *curvature* of a given curve at the point A is the limit of the average curvature along the arc AB when $B \to A$, that is,

$$k = \lim_{s \to 0} (\varphi/s).$$

The curvature of a circle

$$k_{\text{cir}} = 1/a,$$

where a is the radius of the circle; the curvature of a straight line is equal to zero.

If the curve is specified by the equation $y = f(x)$, its curvature can be calculated by the formula

$$k = \frac{|y''|}{(1 + y'^2)^{3/2}}.$$

If the curve is specified by the parametric equations $x = \varphi(t)$, $y = \psi(t)$, then

$$k = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}},$$

where $\dot{x} = \dfrac{dx}{dt}$, $\dot{y} = \dfrac{dy}{dt}$, $\ddot{x} = \dfrac{d^2x}{dt^2}$, $\ddot{y} = \dfrac{d^2y}{dt^2}$.

If the curve is specified by the equation $\rho = f(\theta)$ in polar coordinates, then

$$k = \frac{|\rho^2 + 2\rho'^2 - \rho\rho''|}{(\rho^2 + \rho'^2)^{3/2}},$$

where $\rho' = \dfrac{d\rho}{d\theta}$, $\rho'' = \dfrac{d^2\rho}{d\theta^2}$.

The *radius of curvature* is an inverse of the curvature:

$$R = 1/|k|.$$

The *circle of curvature* of a given curve at its point A is the limiting position of the circle passing through three points A, B, C of the curve when $B \to A$, and $C \to A$.

The radius of the circle of curvature is equal to the radius of curvature. The centre of the circle of curvature is called the *centre of curvature* and lies on the normal to the curve drawn at the point A and directed towards the concavity of that curve.

The coordinates ξ and η of the centre of curvature of the curve $y = f(x)$ can be calculated by the formulas

$$\xi = x - \frac{y'(1 + y'^2)}{y''}, \quad \eta = y + \frac{1 + y'^2}{y''}.$$

The *evolute* of the curve is the set of the centres of its curvature. The formulas specifying the coordinates of the centre of curvature can be regarded as the parametric equations of the evolute (where the parameter is the abscissa x of the original curve).

1096. Find the curvature of the curve $y = -x^3$ at the point with the abscissa $x = 1/2$.

Solution. We have $y' = -3x^2$, $y'' = -6x$. For $x = 1/2$ these derivatives assume the values $y' = -3/4$, $y'' = -3$ and

$$k = \left| \frac{-3}{(1 + 9/16)^{3/2}} \right| = \frac{3}{125/64} = \frac{192}{125}.$$

1097. Find the curvature at any point of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

Solution. We find

$$\dot{x} = a(1 - \cos t), \quad \ddot{x} = a \sin t, \quad \dot{y} = a \sin t, \quad \ddot{y} = a \cos t,$$

$$\dot{x}\ddot{y} - \dot{y}\ddot{x} = a^2(\cos t - \cos^2 t - \sin^2 t) = -a^2(1 - \cos t),$$

$$\dot{x}^2 + \dot{y}^2 = a^2(1 - 2\cos t + \cos^2 t + \sin^2 t) = 2a^2(1 - \cos t),$$

$$k = \left| \frac{-a^2(1 - \cos t)}{2^{3/2}a^3(1 - \cos t)^{3/2}} \right| = \frac{1}{2^{3/2}a(1 - \cos t)^{1/2}}.$$

1098. Find the coordinates of the centre of curvature of the curve $x^3 + y^4 = 2$ at the point $M(1; 1)$.

Solution. Let us differentiate twice the equation of the given curve:

$$3x^2 + 4y^3 \cdot y' = 0 \; (*),$$

$$6x + 12y^2 \cdot y'^2 + 4y^3 \cdot y''^2 = 0 \; (**).$$

Since $x = 1$, $y = 1$, we find $y' = -3/4$ from equation $(*)$ and get $6 + 27/4 + 4y'' = 0$ from equation $(**)$, i.e. $y'' = -51/16$. Then

$$\xi = x - \frac{(1 + y'^2)y'}{y''} = 1 - \frac{(1 + 9/16)(-3/4)}{-51/16} = \frac{43}{68},$$

$$\eta = y + \frac{1 + y'^2}{y''} = 1 + \frac{1 + 9/16}{-51/16} = \frac{26}{51},$$

i.e. $C(43/68; 26/51)$.

1099. Derive the equation of the evolute of the parabola $2y^2 = 2x + 1$.

Solution. We differentiate twice the equation of the parabola

$$4yy' = 2, \quad y' = \frac{1}{2y}; \quad 4y'^2 + 4yy'' = 0, \quad y'' = -\frac{y'^2}{y} = -\frac{1}{4y^3}.$$

Next we determine the coordinates of the centre of curvature:

$$\xi = x - \frac{(1 + y'^2) \cdot y'}{y''} = y^2 - \frac{1}{2} - \frac{\left(1 + \frac{1}{4y^2}\right) \cdot \frac{1}{2y}}{-1/(4y^3)} = 3y^2,$$

$$\eta = y + \frac{1 + y'^2}{y''} = y + \frac{1 + \dfrac{1}{4y^2}}{-1/(4y^3)} = y - 4y^3 - y = -4y^3.$$

We obtain the equation of the evolute in parametric form: $\xi = 3y^2, \eta = -4y^3$. Eliminating the parameter y, we find the equation of the evolute in explicit form: $\eta^2 = 16\xi^3/27$.

1100. Find the radius of curvature of the ellipse $x^2/25 + y^2/9 = 1$ at the point $M(0; 3)$.

1101. Find the radius of curvature at any point of the cardioid $\rho = a(1 + \cos\theta)$ $(a > 0)$.

1102. Find the curvature of the curve $x = e^t \sin t, y = e^t \cos t$ at the point $t = 1$.

1103. Find the coordinates of the centre of curvature of the curve $y = 1/x$ at the point $M(1; 1)$.

1104. Derive the equation of the evolute of the curve $x = t \sin t + \cos t$, $y = \sin t - \sin t$.

7.4. The Order of Contact of Plane Curves

If the curves $y = f(x)$ and $y = \varphi(x)$ have a common point $M(x_0; y_0)$, i.e. $y_0 = f(x_0) = \varphi(x_0)$, and the tangents to these curves drawn at the point $M(x_0; y_0)$ do not coincide, then it is said that the curves $y = f(x)$ and $y = \varphi(x)$ *intersect* at the point M. Here is the condition under which these curves intersect at the point $M(x_0; y_0)$:

$$f(x_0) = \varphi(x_0),\ f'(x_0) \neq \varphi'(x_0).$$

Now if these curves possess a common point $M(x_0; y_0)$ and the tangents to the two curves at this point coincide, then it is said that the curves *touch each other* or *have contact* at the point M. The condition under which the curves have contact at the point $M(x_0; y_0)$ is the following:

$$f(x_0) = \varphi(x_0), f'(x_0) = \varphi'(x_0).$$

If, finally,

$$f(x_0) = \varphi(x_0),\quad f'(x_0) = \varphi'(x_0),\quad f''(x_0) = \varphi''(x_0), \dots, f^{(n)}(x_0) = \varphi^{(n)}(x_0),$$

but $f^{(n+1)}(x_0) \neq \varphi^{(n+1)}(x_0)$, then it is customary to say that at the point $M(x_0; y_0)$ the curves $y = f(x)$ and $y = \varphi(x)$ have *contact of the nth order*.

If $n \geqslant 2$, then at the point $M(x_0; y_0)$ the curves $y = f(x)$ and $y = \varphi(x)$ have not only a tangent in common but also the same curvature.

1105. What is the order of contact of the curves $y = e^{-x}$ and $xy = 1/e$ at the point $x = 1$?

Solution. Assume $f(x) = e^{-x}$, $\varphi(x) = 1/(ex)$. Find the consecutive derivatives of these functions: $f'(x) = -e^{-x}$, $f''(x) = e^{-x}$, $\varphi'(x) = -1/(ex^2)$, $\varphi''(x) = 2/(ex^3)$, ... Now compute the values of the given functions and their

derivatives at the point $x = 1$; you get $f(1) = e^{-1}, f'(1) = -e^{-1}, f''(1) = e^{-1}$; $\varphi(1) = e^{-1}$, $\varphi'(1) = -e^{-1}$, $\varphi''(1) = 2e^{-1}$. Thus it follows that $f(1) = \varphi(1)$, $f'(1) = \varphi'(1)$, but $f''(1) \neq \varphi''(1)$. Consequently, the indicated curves have contact of the first order.

1106. At what choice of the parameter a does the curve $y = e^{ax}$ have, at the point $x = 0$, contact of the first order with the straight line $y = 2x + 1$?

Solution. Assume $f(x) = e^{ax}$ and $\varphi(x) = 2x + 1$. For the indicated lines to have contact of the first order at the point $x = 0$, it is necessary that $f(0) = \varphi(0)$ and $f'(0) = \varphi'(0)$, i.e. $e^{a \cdot 0} = 2 \cdot 0 + 1$ and $ae^0 = 2$. Hence $a = 2$.

1107. What is the order of contact of the curves $y = 1 + \cos x$ and $y = 2 - x^2$ at the point $x = 0$?

1108. What is the order of contact of the curve $y = \sin^2 x$ and the x-axis at the point $x = 0$?

1109. What is the order of contact of the catenary $y = (e^x + e^{-x})/2$ and the parabola $y = 1 + x^2/2$ at the point $x = 0$?

1110. What is the order of contact of the circles $x^2 + y^2 = 2y$ and $x^2 + y^2 = 4y$ at the point $x = 0$?

1111. What is the order of contact of the parabola $y = x^4$ and the x-axis at the point $x = 0$?

1112. What is the order of contact of the curve $y = \ln(1 + x)$ and the parabola $y = x - x^2$ at the point $x = 0$?

7.5. Vector Function of a Scalar Argument and Its Derivative

A space curve can be specified by the parametric equations

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

or by the vector equation

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

The last equation defines the variable vector $\mathbf{r}$ as the *vector function* of the scalar argument t, i.e. $\mathbf{r} = \mathbf{r}(t)$. The curve specified by the equation $\mathbf{r} = \mathbf{r}(t)$ is called a *hodograph* of the variable vector $\mathbf{r}$.

The *derivative* for the vector function $\mathbf{r} = \mathbf{r}(t)$ with respect to the scalar argument t is a new vector function specified by the equation

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t}.$$

The derivative of the vector function can be calculated by the formula

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}.$$

The derivative $d\mathbf{r}/dt$ is a vector directed along the tangent to the hodograph of the vector $\mathbf{r}$ and pointing in the direction in which the increase of the parameter t occurs.

If t is time, then $d\mathbf{r}/dt$ is the velocity vector of the end point of the vector $\mathbf{r}$, and $d^2\mathbf{r}/dt^2$ is the acceleration vector.

The main rules for differentiating a vector function of a scalar argument are the following:

1°. $\dfrac{d}{dt}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3) = \dfrac{d\mathbf{r}_1}{dt} + \dfrac{d\mathbf{r}_2}{dt} - \dfrac{d\mathbf{r}_3}{dt}$;

2°. $\dfrac{d\mathbf{c}}{dt} = 0$, where $\mathbf{c}$ is a constant vector;

3°. $\dfrac{d}{dt}(\lambda\mathbf{r}) = \lambda\dfrac{d\mathbf{r}}{dt} + \mathbf{r}\dfrac{d\lambda}{dt}$, where $\lambda = \lambda(t)$ is a scalar function of t;

4°. $\dfrac{d}{dt}(\mathbf{r}_1 \cdot \mathbf{r}_2) = \dfrac{d\mathbf{r}_1}{dt} \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot \dfrac{d\mathbf{r}_2}{dt}$;

5°. $\dfrac{d}{dt}(\mathbf{r}_1 \times \mathbf{r}_2) = \dfrac{d\mathbf{r}_1}{dt} \times \mathbf{r}_2 + \mathbf{r}_1 \times \dfrac{d\mathbf{r}_2}{dt}$.

The *equations of the tangent* to the space curve $\mathbf{r} = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ at the point $M_0(x_0; y_0; z_0)$ are written in the form

$$(x - x_0)/\dot{x}_0 = (y - y_0)/\dot{y}_0 = (z - z_0)/\dot{z}_0,$$

where $x_0 = x(t_0), y_0 = y(t_0), z_0 = z(t_0), \dot{x}_0 = x'(t_0), \dot{y}_0 = y'(t_0), \dot{z}_0 = z'(t_0)$.

A *normal plane* is a plane passing through the point of contact at right angles to the tangent line. The *equation of a normal plane* has the form

$$\dot{x}_0(x - x_0) + \dot{y}_0(y - y_0) + \dot{z}(z - z_0) = 0.$$

The *differential of the arc of a space curve* can be calculated by the formula

$$ds = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,dt.$$

1113. What curve is the hodograph of the vector function $\mathbf{r} = a\mathbf{i}\cos t + a\mathbf{j}\sin t + ct\mathbf{k}$?

Solution. The curve has the parametric equations $x = a\cos t$, $y = a\sin t$, $z = ct$ specifying a helical line.

1114. What curve is the hodograph of the vector function $\mathbf{r} = \mathbf{i}\cos t + \mathbf{j} + \mathbf{k}\sin t$?

1115. What curve is the hodograph of the vector function $\mathbf{r} = t(\mathbf{i} + \mathbf{j} + \mathbf{k})$?

1116. What curve is the hodograph of the vector function $\mathbf{r} = \mathbf{i} + \mathbf{j} + t\mathbf{k}$?

1117. What curve is the hodograph of the vector function $\mathbf{r} = \mathbf{i}\cosh t + \mathbf{k}\sinh t$?

1118. Find the derivative of the scalar product of the vectors $\mathbf{r}_1 = 3t\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$ and $\mathbf{r}_2 = 2\mathbf{i} - 3t\mathbf{j} + \mathbf{k}$.

Solution. We have

$$\frac{d(\mathbf{r}_1 \cdot \mathbf{r}_2)}{dt} = \mathbf{r}_1 \cdot \frac{d\mathbf{r}_2}{dt} + \mathbf{r}_2\frac{d\mathbf{r}_1}{dt}$$

$$= (3t\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}) \cdot (-3\mathbf{j}) + (2\mathbf{i} - 3t\mathbf{j} + \mathbf{k}) \cdot 3\mathbf{i} = -6 + 6 = 0.$$

The result can be explained by the fact that the scalar product $\mathbf{r}_1 \cdot \mathbf{r}_2 = 5$, that is, it is a constant quantity.

1119. Show that the vectors $\mathbf{r} = \mathbf{i}\cos t + \mathbf{j}\sin t + \mathbf{k}$ and $d\mathbf{r}/dt$ are perpendicular.

Solution. We have $d\mathbf{r}/dt = -\mathbf{i}\sin t + \mathbf{j}\cos t$. We find the scalar product:

$$\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = -\cos t \cdot \sin t + \sin t \cdot \cos t + 1 \cdot 0 = 0.$$

Consequently, $\mathbf{r} \perp \dfrac{d\mathbf{r}}{dt}$.

1120. Find the derivative of the vector function $\mathbf{r} = \mathbf{i}\cosh^2 t + \mathbf{j}\sinh t\cosh t + \mathbf{k}\sinh^2 t$.

1121. $\mathbf{r} = \mathbf{i}\sinh t + \mathbf{j}\cosh t + \mathbf{k}\sqrt{\cosh^2 t - 3\sinh^2 t}$. Find $d(\mathbf{r}^2)/dt$.

1122. $\mathbf{r}_1 = \mathbf{i}t + \mathbf{j}t^2 + \mathbf{k}t^3$, $\mathbf{r}_2 = \mathbf{i}t^2 + \mathbf{j}t^3 + \mathbf{k}t$. Find $d(\mathbf{r}_1 \times \mathbf{r}_2)/dt$.

1123. Derive the equations of the tangent and the normal plane to the curve $x = a\sin^2 t$, $y = b\sin t\cos t$, $z = c\cos^2 t$ at the point $t = \pi/4$.

Solution. We find $\dot{x} = a\sin 2t$, $\dot{y} = b\cos 2t$, $\dot{z} = -c\sin 2t$. For $t = \pi/4$ we have $x_0 = a/2$, $y_0 = b/2$, $z_0 = c/2$, $\dot{x}_0 = a$, $\dot{y}_0 = 0$, $\dot{z}_0 = -c$.

The equation of the tangent is

$$(x - a/2)/a = (y - b/2)/0 = (z - c/2)/(-c).$$

The equation of the normal plane is

$$a\left(x - \frac{a}{2}\right) - c\left(z - \frac{c}{2}\right) = 0, \qquad \text{or} \quad ax - cz - \frac{a^2 - c^2}{2} = 0.$$

1124. Find the equations of the tangent and the normal plane to the helical line $\mathbf{r} = \mathbf{i}\cos t + \mathbf{j}\sin t + \sqrt{3}\,t\mathbf{k}$ at the point $t = \pi/2$.

1125. Find the point of the curve $x = t + 1$, $y = t^2 - 1$, $z = t^3$ at which the tangent is parallel to the plane $x + 2y + z - 1 = 0$.

1126. What angle is formed by the plane xOy and the tangent to the helical line $x = \cos t$, $y = \sin t$, $z = 2\sqrt{2}\,t$ at the point $t = \pi/4$?

1127. Derive the equations of the tangent and the normal plane to the curve $x = (1/\sqrt{2})\,e^t\sin t$, $y = 1$, $z = (1/\sqrt{2})e^t\cos t$ at the point $t = 0$.

1128. Derive the equations of the tangent to the curve $x = e^t(\cos t + \sin t)$, $y = e^t(\sin t - \cos t)$, $z = e^t$ at the point $t = 0$.

1129. Derive the equations of the tangent to the curve $\mathbf{r} = t^2\mathbf{i} + t^3\mathbf{j} + t^4\mathbf{k}$ at the point $t = 1$.

1130. Show that the curves $\mathbf{r} = (u + 1)\mathbf{i} + u^2\mathbf{j} + (2u - 1)\mathbf{k}$ and $\mathbf{r} = 2v^2\mathbf{i} + (3v - 2)\mathbf{j} + v^2\mathbf{k}$ intersect and determine the angle between the curves at the point of their intersection.

1131. Derive the equations of the helical line if the radius of the base of the cylinder $R = 4$, the pitch $h = 6\pi$, and find the differential of its arc.

Solution. The equations of the helical line have the form $x = 4 \cos t$, $y = 4 \sin t$, $z = 3t$ since $z = h$ at $t = 2\pi$. Let us differentiate these equations: $\dot{x} = -4 \sin t, \dot{y} = 4 \cos t, \dot{z} = 3$. Consequently, the differential of the arc is equal to

$$ds = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,dt = \sqrt{16 \sin^2 t + 16 \cos^2 t + 9}\,dt$$
$$= \sqrt{16(\sin^2 t + \cos^2 t) + 9}\,dt = 5dt.$$

1132. Find the differential of the arc of the curve $x = a \cos^2 t$, $y = \sqrt{a^2 + b^2} \sin t \cos t$, $z = b \sin^2 t$.

1133. At what pitch h is the length of the arc of one coil of the helical line $x = \cos t$, $y = \sin t$, $z = ct$ equal to 4π?

Hint. Make use of the fact that when the cylinder is developed on a plane, a coil of a helical line turns into a segment of a straight line.

1134. The equation of motion has the form $\mathbf{r} = 3\mathbf{i} \cos t + 3\mathbf{j} \sin t + 4t\mathbf{k}$, where t is time. Determine the velocity and the acceleration of motion at an arbitrary moment of time.

1135. The equation of motion has the form $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$. Determine the velocity and the acceleration of motion at the moment $t = 1$.

7.6. A Moving Trihedral of a Space Curve. Curvature and Torsion

At any point $M(x; y; z)$ of the space curve $\mathbf{r} = \mathbf{r}(t)$ it is possible to construct three perpendicular unit vectors (Fig. 40): a unit vector of the *tangent* (tangential unit vec-

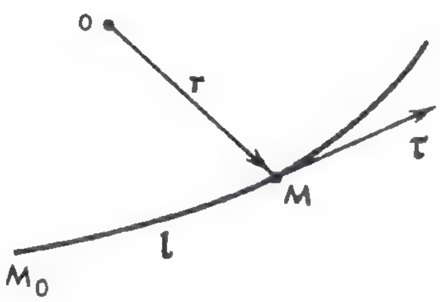

Fig. 40

tor)

$$\tau = \frac{d\mathbf{r}}{ds} = \frac{\dfrac{d\mathbf{r}}{dt}}{\left|\dfrac{d\mathbf{r}}{dt}\right|};$$

a unit vector of the *principle normal*

$$\nu = \frac{\dfrac{d\tau}{ds}}{\left|\dfrac{d\tau}{ds}\right|};$$

a unit vector of the *binormal*

$$\beta = \tau \cdot \nu.$$

The corresponding nonunit vectors can be found by the formulas

$$\mathbf{T} = \frac{d\mathbf{r}}{dt} \text{ (the vector of the tangent),}$$

$$\mathbf{N} = \mathbf{B} \times \mathbf{T} \text{ (the vector of the principle normal),}$$

$$\mathbf{B} = \frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2} \text{ (the vector of the binormal).}$$

The plane containing the vectors τ and ν is called an *osculating* plane; that containing the vectors ν and β is called a *normal* plane, and that containing the vectors β and τ is called a *rectifying* plane.

A trihedral with the vertex at the point M formed by the osculating, normal and rectifying planes is known as a *moving trihedral* of a space curve (Fig. 41).

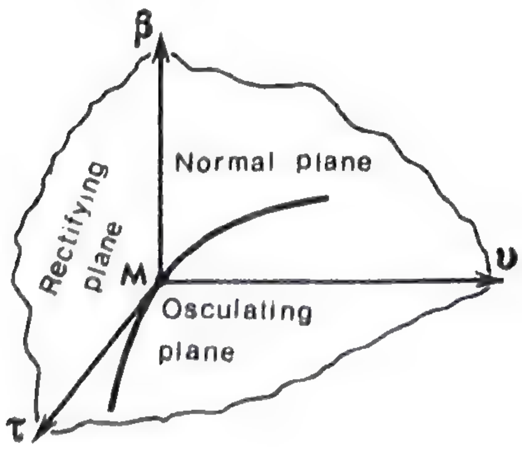

Fig. 41

The *curvature* of a curve at the point M is the number

$$K = \lim_{\Delta s \to 0} \frac{\varphi}{\Delta s},$$

where φ is the angle of rotation of the tangent line (the angle of contingence) along the arc MN, Δs is the length of that arc.

If a curve is specified by the equation $\mathbf{r} = \mathbf{r}(s)$, then $K = \left|\frac{d\tau}{ds}\right|$.

If the equation of a curve has the form $\mathbf{r} = \mathbf{r}(t)$, then

$$K = \frac{\left|\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}\right|}{\left|\frac{d\mathbf{r}}{dt}\right|^3}.$$

The *torsion* of a curve at the point M is the number

$$\sigma = \lim_{\Delta s \to 0} \frac{\theta}{\Delta s},$$

where θ is the angle of rotation of the binormal (the angle of contingence of the 2nd kind) along the arc MN.

If $\mathbf{r} = \mathbf{r}(s)$, then $\sigma = \mp\left|\frac{d\beta}{ds}\right|$, where the minus sign is taken in the case of the like directions of the vectors $\frac{d\beta}{ds}$ and ν and the plus sign in the case of their opposite directions.

If $\mathbf{r} = \mathbf{r}(t)$, then

$$\sigma = \frac{\frac{d\mathbf{r}}{dt}\,\frac{d^2\mathbf{r}}{dt^2}\,\frac{d^3\mathbf{r}}{dt^3}}{\left|\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt}\right|^2}.$$

1136. Find the tangential vector τ of the curve $\mathbf{r} = t^2\mathbf{i} + t^3\mathbf{j} + t^6\mathbf{k}$ at the point $t = 1$.

Solution. We have

$$\frac{d\mathbf{r}}{dt} = 2t\mathbf{i} + 3t^2\mathbf{j} + 6t^5\mathbf{k}, \quad \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{4t^2 + 9t^4 + 36t^{10}}.$$

At $t = 1$ we find

$$\frac{d\mathbf{r}}{dt} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}, \quad \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{4 + 9 + 36} = 7,$$

$$\tau = \frac{\frac{d\mathbf{r}}{dt}}{\left|\frac{d\mathbf{r}}{dt}\right|} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

1137. Find the tangential unit vector of the curve $\mathbf{r} = 5t\mathbf{i} + 12\mathbf{j}\cos t + 12\mathbf{k}\sin t$ at an arbitrary point.

1138. Find the tangential unit vector of the curve $x = t\sin t + \cos t, y = t\cos t - \sin t, x = t^2\sqrt{2}$ at the point $t = \pi/2$.

1139. Find the vector τ of the helical line $x = a\cos t, y = a\sin t, z = \sqrt{R^2 - a^2}\,t$, $R > a > 0$ at an arbitrary point.

Solution. We have

$$\mathbf{r} = a\mathbf{i}\cos t + a\mathbf{j}\sin t + \sqrt{R^2 - a^2}\,t\mathbf{k},$$

$$\frac{d\mathbf{r}}{dt} = -a\mathbf{i}\sin t + a\mathbf{j}\cos t + \sqrt{R^2 - a^2}\,\mathbf{k},$$

$$\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + R^2 - a^2} = R,$$

$$\tau = \frac{\frac{d\mathbf{r}}{dt}}{\left|\frac{d\mathbf{r}}{dt}\right|} = \frac{-a\mathbf{i}\sin t + a\mathbf{j}\cos t + \sqrt{R^2 - a^2}\,\mathbf{k}}{R}$$

$$= -\frac{a\sin t}{R}\mathbf{i} + \frac{a\cos t}{R}\mathbf{j} + \frac{\sqrt{R^2 + a^2}}{R}\mathbf{k}.$$

1140. Find the vector β of a helical line at an arbitrary point.

Solution. We have

$$\frac{d\mathbf{r}}{dt} = -a\mathbf{i}\sin t + a\mathbf{j}\cos t + \sqrt{R^2 - a^2}\mathbf{k}, \quad \frac{d^2\mathbf{r}}{dt^2} = -a\mathbf{i}\cos t - a\mathbf{j}\sin t.$$

We find the vector product of these vectors:

$$\mathbf{B} = \frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & \sqrt{R^2 - a^2} \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= a\sqrt{R^2 - a^2}\sin t \cdot \mathbf{i} - a\sqrt{R^2 - a^2}\cos t \cdot \mathbf{j} + a^2\mathbf{k};$$

$$|\mathbf{B}| = \left|\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}\right| = \sqrt{a^2(R^2 - a^2)\sin^2 t + a^2(R^2 - a^2)\cos^2 t + a^4} = aR.$$

Consequently,

$$\beta = \frac{\mathbf{B}}{|\mathbf{B}|} = \frac{a\sqrt{R^2 - a^2}\,\mathbf{i}\sin t - a\sqrt{R^2 - a^2}\,\mathbf{j}\cos t + a^2\mathbf{k}}{aR}$$

$$= \frac{\sqrt{R^2 - a^2}}{R}\sin t \cdot \mathbf{i} - \frac{\sqrt{R^2 - a^2}}{R}\cos t \cdot \mathbf{j} + \frac{a}{R}\mathbf{k}.$$

1141. Find the vector ν of a helical line at an arbitrary point.
Solution. Since $\nu = \beta \times \tau$, we have

$$\nu = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\sqrt{R^2 - a^2}\sin t}{R} & -\dfrac{\sqrt{R^2 - a^2}\cos t}{R} & \dfrac{a}{R} \\ -\dfrac{a\sin t}{R} & \dfrac{a\cos t}{R} & \dfrac{\sqrt{R^2 - a^2}}{R} \end{vmatrix} = -\mathbf{i}\cos t - \mathbf{j}\sin t.$$

1142. Find the curvature K of a helical line.

Solution. We have found in problems 1139 and 1140 that $\left|\dfrac{d\mathbf{r}}{dt}\right| = R$, $\left|\dfrac{d\mathbf{r}}{dt} \times \dfrac{d^2\mathbf{r}}{dt^2}\right| = aR$, therefore

$$K = \frac{\left|\dfrac{d\mathbf{r}}{dt} \times \dfrac{d^2\mathbf{r}}{dt^2}\right|}{\left|\dfrac{d\mathbf{r}}{dt}\right|^3} = \frac{aR}{R^3} = \frac{a}{R^2}.$$

1143. Find the torsion σ of a helical line.
Solution. We have

$$\frac{d\mathbf{r}}{dt} = -a\mathbf{i}\sin t + a\mathbf{j}\cos t + \sqrt{R^2 - a^2}\,\mathbf{k},$$

$$\frac{d^2\mathbf{r}}{dt^2} = -a\mathbf{i}\cos t - a\mathbf{j}\sin t, \quad \frac{d^3\mathbf{r}}{dt^3} = a\mathbf{i}\sin t - a\mathbf{j}\cos t.$$

We find the mixed product of these vectors:

$$\frac{d\mathbf{r}}{dt}\,\frac{d^2\mathbf{r}}{dt^2}\,\frac{d^3\mathbf{r}}{dt^3} = \begin{vmatrix} -a\sin t & a\cos t & \sqrt{R^2-a^2} \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix} = a^2\sqrt{R^2-a^2}\,.$$

We have found in problem 1140 that $\left|\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}\right| = aR$.

Consequently, $\left|\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}\right|^2 = a^2R^2$. Thus we have

$$\sigma = \frac{a^2\sqrt{R^2-a^2}}{a^2R^2} = \frac{\sqrt{R^2-a^2}}{R^2}\,.$$

1144. Derive the equation of an osculating plane of a helical line at an arbitrary point.

Solution. The plane passes through the point $(a\cos t;\ a\sin t;\ \sqrt{R^2-a^2}\,t)$ and is perpendicular to the vector of the binormal $\boldsymbol{\beta} = \frac{\sqrt{R^2-a^2}\sin t}{R}\,\mathbf{i} - \frac{\sqrt{R^2-a^2}\,\cos t}{R}\,\mathbf{j} + \frac{a}{R}\,\mathbf{k}$. Therefore, the equation of the osculating plane is this:

$$\frac{\sqrt{R^2-a^2}}{R}\sin t\,(X - a\cos t) - \frac{\sqrt{R^2-a^2}\,\cos t}{R}\,(Y - a\sin t) + \frac{a}{R}(Z - \sqrt{R^2-a^2}\,t) = 0,$$

or this:

$$X\cdot\sqrt{R^2-a^2}\sin t - Y\cdot\sqrt{R^2-a^2}\cos t + aZ - a\sqrt{R^2-a^2}\,t = 0.$$

1145. Derive the equation of a rectifying plane of a helical line at an arbitrary point.

Solution. The plane passes through the point $(a\cos t;\ a\sin t;\ \sqrt{R^2-a^2}\,t)$ at right angles to the vector of the principal normal $\boldsymbol{\nu} = -\mathbf{i}\cos t - \mathbf{j}\sin t$. Therefore, the equation in question has the form

$$-(X - a\cos t)\cos t - (Y - a\sin t)\sin t = 0, \text{ i.e. } X\cos t + Y\sin t - a = 0.$$

1146. Derive the equation of a normal plane of a helical line at an arbitrary point.

Solution. The plane is perpendicular to the vector of the tangent line $\boldsymbol{\tau} = -\frac{a\sin t}{R}\,\mathbf{i} + \frac{a\cos t}{R}\,\mathbf{j} + \frac{\sqrt{R^2-a^2}}{R}\,\mathbf{k}$ and passes through the point $(a\cos t;\ a\sin t;\ \sqrt{R^2-a^2}\,t)$. Therefore, the desired equation has the form

$$-\frac{a\sin t}{R}(X - a\cos t) + \frac{a\cos t}{R}(Y - a\sin t)$$

$$+ \frac{\sqrt{R^2 - a^2}}{R} (Z - \sqrt{R^2 - a^2}t) = 0,$$

or

$$Xa \sin t - Ya \cos t - Z\sqrt{R^2 - a^2} + (R^2 - a^2)t = 0.$$

1147. Find the vector τ of the curve $x = 6t$, $y = 3t^2$, $z = t^3$ at the point $t = 1$.
1148. Find the vector β of the same curve at $t = 1$.
1149. Find the vector ν of the same curve at $t = 1$.
1150. Find the curvature K of the same curve at $t = 1$.
1151. Find the torsion σ of the same curve at $t = 1$.
1152. Derive the equation of the osculating plane of the same curve at $t = 1$.
1153. Derive the equation of the rectifying plane of the same curve at $t = 1$.
1154. Derive the equation of the normal plane of the same curve at $t = 1$.

Chapter 8

Differential Calculus of Functions of Several Independent Variables

8.1. Domain of Definition of a Function. Level Curves and Surfaces

Assume we are given two nonempty sets D and U. If in accordance with a definite rule each pair of real numbers $(x; y)$ belonging to the set D is associated with one and only one element u from U, then it is said that a *function* f (or a *projection*) having a set of values U is given on the set D. This is written as $D \xrightarrow{f} U$, or f: $D \to U$. The set D is called the *domain of definition* of the function, and the set U consisting of all numbers of the form $f(x, y)$, where $(x; y) \in D$, is called the *range* of the function. The value of the function $u = f(x, y)$ at the point $M(x_0; y_0)$ is designated as $f(x_0, y_0)$ or $f(M)$.

In the simplest cases, the domain of definition of the function $u = f(x, y)$ is either a part of a plane bounded by a closed curve, with the points of the curve (of the boundary of the domain) belonging or not belonging to the domain of definition, or the entire plane, or, finally, a collection of several parts of the xOy plane. The geometric representation of the function $u = f(x, y)$ in the rectangular system of coordinates $Oxyu$ (the graph of the function) is a certain surface.

A similar definition holds for a function of any number of variables $u = f(x, y, z, \ldots, t)$.

The *level curve* of the function $u = f(x, y, z)$ is a curve $f(x, y) = C$ on the xOy plane at whose points the function retains the constant value $u = C$.

The *level surface* of the function $u = f(x, y, z)$ is a surface $f(x, y, z) = C$ at whose points the function retains the constant value $u = C$.

1155. Find the domain of definition of the function $u = \sqrt{a^2 - x^2 - y^2}$.

Solution. The function u assumes real values under the condition $a^2 - x^2 - y^2 \geqslant 0$, or $x^2 + y^2 \leqslant a^2$, that is, the domain of definition of the given function is a circle of radius a with centre at the origin, the boundary circumference inclusive.

1156. Find the domain of definition of the function $u = \arcsin(x/y^2)$.

Solution. The function is defined if $y \neq 0$ and $-1 \leqslant x/y^2 \leqslant 1$, i.e. $-y^2 \leqslant x \leqslant y^2$. The domain of definition of the function is a part of the plane contained between two parabolas $y^2 = x$ and $y^2 = -x$, except for the point $O(0; 0)$.

1157. Find the domain of definition of the function $u = \ln(2z^2 - 6x^2 - 3y^2 - 6)$.

Solution. The given function depends on three variables and assumes real values for $2z^2 - 6x^2 - 3y^2 - 6 > 0$, or $x^2/1 + y^2/2 - z^2/3 < -1$, that is, the domain of definition of the function is a part of space contained inside the sheets of a two-sheet hyperboloid.

1158. Find the level curves of the function $u = x^2 + y^2$.

Solution. The equation of the family of the level curves has the form $x^2 + y^2 = C$ $(C > 0)$. Assigning to C various real values, we obtain concentric circles with centre at the origin.

1159. Find the level surfaces of the function $u = x^2 + z^2 - y^2$.

Solution. The equation of the family of the level surfaces has the form $x^2 + z^2 - y^2 = C$. If $C = 0$, we get $x^2 + z^2 - y^2 = 0$, a cone; if $C > 0$, then $x^2 + z^2 - y^2 = C$, a family of one-sheet hyperboloids; if $C < 0$, then $x^2 + z^2 - y^2 = C$, a family of two-sheet hyperboloids.

Find the domains of definition of the following functions:

1160. $u = \sqrt{x^2 + y^2 - 1}$. **1161.** $u = 1/\sqrt{1 - x^2 - y^2}$.
1162. $u = \arcsin(x + y)$. **1163.** $u = \sqrt{\cos(x^2 + y^2)}$.
1164. $u = \ln(-x + y)$. **1165.** $u = y + \sqrt{x}$.
1166. $u = \sqrt{a^2 - x^2 - y^2 - z^2}$. **1167.** $u = \arcsin(z/\sqrt{x^2 + y^2})$.
1168. $u = 1/\ln(1 - x^2 - y^2 - z^2)$. **1169.** $u = \sqrt{x + y + z}$.

Find the level curves of the functions:

1170. $z = 2x + y$. **1171.** $z = x/y$. **1172.** $z = \ln\sqrt{y/x}$.
1173. $z = \sqrt{x}/y$. **1174.** $z = e^{xy}$.

Find the level surfaces of the functions:

1175. $u = x + y + 3z$. **1176.** $u = x^2 + y^2 + z^2$.
1177. $u = x^2 - y^2 - z^2$.

8.2. Derivatives and Differentials of the Functions of Several Variables

8.2.1. First-order partial derivatives. The *partial derivative* of the function $z = f(x, y)$ with respect to the independent variable x is the derivative

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = f_x'(x, y),$$

calculated for the constant y.

The partial derivative with respect to y is the derivative

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = f_y'(x, y),$$

calculated for the constant x.

The ordinary rules and formulas for differentiation are valid for partial derivatives.

1178. $u = x^2 - 3xy - 4y^2 - x + 2y + 1$. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

Solution. Assuming y to be a constant quantity, we obtain $\frac{\partial u}{\partial x} = 2x - 3y - 1$.

Considering x to be constant, we find $\frac{\partial u}{\partial y} = -3x - 8y + 2$.

1179. $z = e^{x^2 + y^2}$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution. We have

$$\frac{\partial z}{\partial x} = e^{x^2 + y^2}(x^2 + y^2)'_x = 2xe^{x^2 + y^2},$$
$$\frac{\partial z}{\partial y} = e^{x^2 + y^2}(x^2 + y^2)'_y = 2ye^{x^2 + y^2}.$$

1180. $\rho = u^4\cos^2\varphi$. Find $\frac{\partial \rho}{\partial u}$ and $\frac{\partial \rho}{\partial \varphi}$.

Solution. We have

$$\frac{\partial \rho}{\partial u} = 4u^3\cos^2\varphi,$$
$$\frac{\partial \rho}{\partial \varphi} = u^4 \cdot 2\cos\varphi(-\sin\varphi) = -u^4\sin 2\varphi.$$

1181. Show that the function $z = y\ln(x^2 - y^2)$ satisfies the equation $\frac{1}{x} \cdot \frac{\partial z}{\partial x} + \frac{1}{y} \cdot \frac{\partial z}{\partial y} = \frac{z}{y^2}$.

Solution. We find

$$\frac{\partial z}{\partial x} = \frac{2xy}{x^2 - y^2}, \quad \frac{\partial z}{\partial y} = \ln(x^2 - y^2) - \frac{2y^2}{x^2 - y^2}.$$

Substituting the expressions obtained into the left-hand side of the equation

$$\frac{1}{x} \cdot \frac{2xy}{x^2 - y^2} + \frac{1}{y}\left[\ln(x^2 - y^2) - \frac{2y^2}{x^2 - y^2}\right] = \frac{2y}{x^2 - y^2} - \frac{2y}{x^2 - y^2} + \frac{\ln(x^2 - y^2)}{y} \equiv \frac{z}{y^2},$$

we get an identity, that is, the function z satisfies the given equation.

1182. Show that the function $z = y^{y/x}\sin(y/x)$ satisfies the equation $x^2\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = yz$.

Solution. We find

$$\frac{\partial z}{\partial x} = y^{y/x} \cdot \ln y \left(-\frac{y}{x^2}\right) \sin\left(\frac{y}{x}\right) + y^{y/x}\cos\left(\frac{y}{x}\right) \cdot \left(-\frac{y}{x^2}\right),$$

$$\frac{\partial z}{\partial y} = \left[y^{y/x} \cdot \ln y \left(\frac{1}{x}\right) + \frac{y}{x} \cdot y^{y/x - 1}\right] \sin\left(\frac{y}{x}\right) + y^{y/x}\cos\left(\frac{y}{x}\right) \cdot \frac{1}{x}.$$

Substituting the expressions obtained into the left-hand side of the equation

$$-x^2 \frac{y}{x^2} \cdot y^{y/x} \cdot \ln y \cdot \sin\left(\frac{y}{x}\right) - x^2 \cdot \frac{y}{x^2} \cdot y^{y/x}\cos\left(\frac{y}{x}\right)$$
$$+ xy \cdot \frac{y}{x} \cdot y^{y/x - 1} \cdot \sin\left(\frac{y}{x}\right) + xy \cdot \frac{1}{x} \cdot y^{y/x} \cdot \ln y \cdot \sin\left(\frac{y}{x}\right)$$
$$+ xy \cdot \frac{1}{x} \cdot y^{y/x}\cos\left(\frac{y}{x}\right) = y y^{y/x}\sin\left(\frac{y}{x}\right) \equiv yz,$$

we get an identity. Consequently, the function z satisfies the equation.

1183. $u = x^2 + 2y^2 - 3xy - 4x + 2y + 5$. Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$.

1184. $r = \rho^2\sin^4\theta$. Find $\frac{\partial r}{\partial \rho}, \frac{\partial r}{\partial \theta}$.

1185. $u = \frac{x^2}{y^2} - \frac{x}{y}$. Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$.

1186. $z = e^{xy(x^2 + y^2)}$. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

1187. $u = 2y\sqrt{x} + 3y^2\sqrt[3]{z^2}$. Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$.

1188. $u = e^{x/y} + e^{-z/y}$. Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$.

1189. $z = \arctan \frac{y}{1 + x^2}$. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

1190. $z = e^{(x^3 + y^2)^2}$. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

1191. $u = (x - y)(x - z)(y - z)$. Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$.

1192. $u = e^{3x^2 + 2y^2 - xy}$. Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$.

1193. $u = e^{xyz} \cdot \sin\frac{y}{x}$. Find $\frac{\partial u}{\partial y}$.

1194. Show that the function $z = \frac{x^2}{2y} + \frac{x}{2} + \frac{1}{x} - \frac{1}{y}$ satisfies the equation $x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = \frac{x^3}{y}$.

1195. Find $\begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \theta} \end{vmatrix}$, if $x = \rho\cos\theta$, $y = \rho\sin\theta$.

8.2.2. Total differential. The *total increment* of the function $z = f(x, y)$ at the point $M(x; y)$ is the difference $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$, where Δx and Δy are arbitrary increments of the arguments.

The function $z = f(x, y)$ is said to be *differentiable* at the point $(x; y)$ if the total increment at that point can be represented in the form

$$\Delta z = A\Delta x + B\Delta y + o(\rho),$$

where $\rho = \sqrt{\Delta x^2 + \Delta y^2}$.

The *total*, or *exact*, *differential* of the function $z = f(x, y)$ is the principal part of the total increment Δz, which is linear with respect to the increments of the arguments Δx and Δy, i.e. $dz = A\Delta x + B\Delta y$.

The differentials of independent variables coincide with their increments, i.e. $dx = \Delta x$ and $dy = \Delta y$.

The total differential of the function $z = f(x, y)$ can be calculated by the formula

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}\partial y.$$

Similarly, the total differential of the function of three arguments $u = f(x, y, z)$ can be calculated by the formula

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz.$$

For a sufficiently small $\rho = \sqrt{\Delta x^2 + \Delta y^2}$, the approximate equalities

$$\Delta z \approx dz; \quad f(x + \Delta x, \quad y + \Delta y) \approx f(x, y) + dz$$

hold true for the differentiable function $z = f(x, y)$.

1196. $z = \arctan \dfrac{x + y}{x - y}$. Find dz.

Solution. We find the partial derivatives:

$$\frac{\partial z}{\partial x} = \frac{1}{1 + \left(\dfrac{x + y}{x - y}\right)^2} \cdot \frac{-2y}{(x - y)^2} = -\frac{y}{x^2 + y^2},$$

$$\frac{\partial z}{\partial y} = \frac{1}{1 + \left(\dfrac{x + y}{x - y}\right)^2} \cdot \frac{2x}{(x - y)^2} = \frac{x}{x^2 + y^2}.$$

Consequently,

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{xdy - ydx}{x^2 + y^2}.$$

1197. $u = x^{y^2z}$. Find du.

Solution. We have $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$, where

$$\frac{\partial u}{\partial x} = y^2z\cdot x^{y^2z-1}, \quad \frac{\partial u}{\partial y} = x^{y^2z}\cdot \ln x\cdot 2yz, \quad \frac{\partial u}{\partial z} = x^{y^2z}\cdot \ln x\cdot y^2.$$

Consequently,

$$du = y^2zx^{y^2z-1}dx + 2yz\cdot x^{y^2z}\cdot \ln x\, dy + y^2x^{y^2z}\cdot \ln x\, dz.$$

1198. Compute approximately $\sqrt{\sin^2 1.55 + 8e^{0.015}}$, proceeding form the value of the function $z = \sqrt{\sin^2 x + 8e^y}$ at $x = \pi/2 \approx 1.571$, $y = 0$.

Solution. The desired number is the increased value of the function z at $\Delta x = 0.021$, $\Delta y = 0.015$. Let us find the value of z at $x = \pi/2$, $y = 0$. We get $z = \sqrt{\sin^2(\pi/2) + 8e^0} = 3$. Next we find the increment of the function:

$$\Delta z \approx dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y = \frac{\sin 2x\Delta x + 8e^y\Delta y}{2\sqrt{\sin^2 x + 8e^y}} = \frac{8\cdot 0.015}{6} = 0.02.$$

Consequently, $\sqrt{\sin^2 1.55 + 8e^{0\cdot 0.15}} \approx 3.02$.

1199. Compute approximately $\arctan(1.02/0.95)$ proceeding from the value of the function $z = \arctan(y/x)$ at $x = 1$, $y = 1$.

Solution. The value of the function z at $x = 1$, $y = 1$ is $z = \arctan(1/1) = \pi/4 \approx 0.785$. Let us find the increment of the function Δz at $\Delta x = -0.05$, $\Delta y = 0.02$:

$$\Delta z \approx dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y = -\frac{y\Delta x}{x^2+y^2} + \frac{x\Delta y}{x^2+y^2}$$
$$= \frac{x\Delta y - y\Delta x}{x^2+y^2} = \frac{1\cdot 0.02 + 1\cdot 0.05}{2} = 0.035.$$

Consequently, $\arctan(1.02/0.95) = z + \Delta z \approx 0.785 + 0.035 = 0.82$.

1200. $z = \ln(x^2 + y^2)$. Find dz. **1201.** $z = \ln\tan(y/x)$. Find dz.

1202. $z = \sin(x^2 + y^2)$. Find dz. **1203.** $z = x^y$. Find dz.

1204. $u = \ln(x + \sqrt{x^2 + y^2})$. Find du. **1205.** $z = e^x(\cos y + x\sin y)$. Find dz.

1206. $z = e^{x+y}(x\cos y + y\sin x)$. Find dz.

1207. $z = \arctan\dfrac{2(x + \sin y)}{4 - x\sin y}$. Find dz.

1208. $u = e^{xyz}$. Find du.

1209. Compute approximately $1.02^{4.05}$ proceeding from the value of the function $z = x^y$ at $x = 1$, $y = 4$ and replacing its increment by the differential.

1210. Compute approximately $\ln(0.09^3 + 0.99^3)$ proceeding from the value of the function $z = \ln(x^3 + y^3)$ at $x = 0$, $y = 1$.

1211. Compute approximately $\sqrt[3]{1.02^2 + 0.05^2}$ proceeding from the value of the function $z = \sqrt[3]{x^2 + y^2}$ at $x = 1$, $y = 0$.

1212. Compute approximately $\sqrt{5e^{0.02} + 2.03^2}$ proceeding from the value of the function $z = \sqrt{5e^x + y^2}$ at $x = 0, y = 2$.

1213. Compute approximately $\sqrt{1.04^{1.99} + \ln 1.02}$ proceeding from the value of the function $u = \sqrt{x^y + \ln z}$ at $x = 1, y = 2, z = 1$.

8.2.3. Partial derivatives and differentials of higher orders. The *second-order partial derivatives* of the function $z = f(x, y)$ are the partial derivatives of its first-order partial derivatives.

This is how the second-order partial derivatives are designated:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial^2 x} = f''_{xx}(x, y); \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x \partial y} = f''_{xy}(x, y);$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y \partial x} = f''_{yx}(x, y); \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f''_{yy}(x, y).$$

Analogous designations are used for the third- and the higher-order derivatives. For instance,

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial^3 z}{\partial x^3} = f'_{xxx}(x, y); \quad \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial^3 z}{\partial x^2 \partial y} = f'''_{xxy}(x, y), \text{ etc.}$$

are the so-called mixed derivatives differing from one another only by the order of differentiation.. They equal to each other in the case they are cointinuous, for example, $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.

The *second-order differential* of the function $z = f(x, y)$ is the differential of its total differential, i.e. $d^2 z = d(dz)$.

The differentials of the third- and the higher-orders are defined by analogy: $d^3 z = d(d^2 z)$; in general, $d^n z = d(d^{n-1} z)$.

If x and y are independent variables and the functions $f(x, y)$ possesses continuous partial derivatives, then the differentials of the higher orders can be calculated by the formulas

$$d^2 z = \frac{\partial^2 z}{\partial x^2} dx^2 + 2\frac{\partial^2 z}{\partial x \partial y} dx dy + \frac{\partial^2 z}{\partial y^2} dy^2;$$

$$d^3 z = \frac{\partial^3 z}{\partial x^3} dx^3 + 3\frac{\partial^3 z}{\partial x^2 \partial y} dx^2 dy + 3\frac{\partial^3 z}{\partial x \partial y^2} dx dy^2 + \frac{\partial^3 z}{\partial y^3} dy^3.$$

In a general case, there holds a symbolic formula

$$d^n z = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy\right)^n z,$$

which can be formally expanded in accordance with the binomial law.

1214. $z = y \ln x$. Find $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$.

Solution. Let us find the partial derivatives:

$$\frac{\partial z}{\partial x} = \frac{y}{x}; \quad \frac{\partial z}{\partial y} = \ln x.$$

Differentiating once again, we get

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = -\frac{y}{x^2}; \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(\ln x) = 0; \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{x}.$$

1215. $z = \ln\tan(y/x)$. Find $\dfrac{\partial^2 z}{\partial x \partial y}$.

Solution. We have

$$\frac{\partial z}{\partial x} = \frac{1}{\tan(y/x)} \cdot \sec^2(y/x)\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2} \cdot \frac{2}{\sin(2y/x)},$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x^2} \cdot \frac{2}{\sin(2y/x)} - \frac{y}{x^2} \cdot \frac{-2\cos(2y/x) \cdot (2/x)}{\sin^2(2y/x)}$$

$$= \frac{2}{x^3 \cdot \sin^2(2y/x)} \cdot (2y\cos(2y/x) - x\sin(2y/x)).$$

1216. $z = \sin x \sin y$. Find $d^2 z$.
Solution. We have

$$\frac{\partial z}{\partial x} = \cos x \sin y, \quad \frac{\partial z}{\partial y} = \sin x \cos y,$$

$$\frac{\partial^2 z}{\partial x^2} = -\sin x \sin y, \quad \frac{\partial^2 z}{\partial x \partial y} = \cos x \cos y, \quad \frac{\partial^2 z}{\partial y^2} = -\sin x \sin y,$$

$$d^2 z = -\sin x \sin y dx^2 + 2\cos x \cos y dx dy - \sin x \sin y dy^2.$$

1217. $z = x^2 y$. Find $d^3 z$.
Solution. We have

$$\frac{\partial z}{\partial x} = 2xy, \quad \frac{\partial^2 z}{\partial x^2} = 2y, \quad \frac{\partial^3 z}{\partial x^3} = 0, \quad \frac{\partial z}{\partial y} = x^2, \quad \frac{\partial^2 z}{\partial y^2} = 0,$$

$$\frac{\partial^3 z}{\partial y^3} = 0, \quad \frac{\partial^3 z}{\partial x^2 \partial y} = 2, \quad \frac{\partial^3 z}{\partial x \partial y^2} = 0;$$

$$d^3 z = 0 \cdot dx^3 + 3 \cdot 2dx^2 dy + 3 \cdot 0 \cdot dx \cdot dy^2 + 0 \cdot dy^3 = 6dx^2 dy.$$

1218. $u = 4x^3 + 3x^2 y + 3xy^2 - y^3$. Find $\dfrac{\partial^2 u}{\partial x \partial y}$.

1219. $u = xy + \sin(x + y)$. Find $\dfrac{\partial^2 u}{\partial x^2}$.

1220. $u = \ln\tan(x + y)$. Find $\dfrac{\partial^2 u}{\partial x \partial y}$.

1221. $z = \arctan \dfrac{x+y}{1-xy}$. Find $\dfrac{\partial^2 z}{\partial x \partial y}$.

1222. $z = x^2 \ln(x+y)$. Find $\dfrac{\partial^2 z}{\partial x \partial y}$.

1223. $u = x\sin\ xy + y\cos\ xy$. Find $\dfrac{\partial^2 u}{\partial x^2}$.

1224. $u = \sin(x + \cos y)$. Find $\dfrac{\partial^3 u}{\partial x^2 \partial y}$.

1225. $z = 0.5 \ln(x^2 + y^2)$. Find d^2z.

1226. $z = \cos(x+y)$. Find d^2z.

1227. $z = \cos(ax + e^y)$. Find $\dfrac{\partial^3 z}{\partial x \partial y^2}$.

1228. $u = \dfrac{x^4 - 8xy^3}{x - 2y}$. Find $\dfrac{\partial^3 u}{\partial x^2 \partial y}$.

1229. $z = x^2y^3$. Verify that $\dfrac{\partial^5 z}{\partial x^2 \partial y^3} = \dfrac{\partial^5 z}{\partial y^3 \partial x^2}$.

1230. $z = x^2 + y^2 - xy - 2x + y + 7$. Find d^2z.

1231. Show that the function $z = \varphi(x)g(y)$ satisfies the equation $z \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial z}{\partial x} \cdot \dfrac{\partial z}{\partial y}$.

1232. Show that the function $z = g(x) + yg'(x)$ satisfies the equation $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y} + y\dfrac{\partial^2 z}{\partial x \partial y}$.

8.2.4. Differentiation of composite functions. Suppose $z = f(x, y)$, where $x = \varphi(t)$, $y = \psi(t)$, and the functions $f(x, y)$, $\varphi(t)$, $\psi(t)$ are differentiable.

Then the derivative of the composite function $z = f[\varphi(t), \psi(t)]$ can be calculated by the formula

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}.$$

If $z = f(x, y)$, where $y = \varphi(x)$, then the *total derivative* of z with respect to x can be found by the formula

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}.$$

Now if $z = f(x, y)$, where $x = \varphi(\xi, \eta)$, $y = \psi(\xi, \eta)$, then the partial derivatives

are expressed as follows:

$$\frac{\partial z}{\partial \xi} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial \xi} \quad \text{and} \quad \frac{\partial z}{\partial \eta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial \eta}.$$

1233. $z = e^{x^2 + y^2}$, where $x = a\cos t$, $y = a\sin t$. Find $\frac{dz}{dt}$.

Solution. We have

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = e^{x^2 + y^2} \cdot 2x(-a\sin t) + e^{x^2 + y^2} \cdot 2y(a\cos t)$$
$$= 2ae^{x^2 + y^2}(y\cos t - x\sin t).$$

Expressing x and y in terms of t, we get

$$\frac{dz}{dt} = 2ae^{a^2}(a\sin t\cos t - a\cos t\sin t) = 0.$$

1234. $z = \ln(x^2 - y^2)$, where $y = e^x$. Find $\frac{\partial z}{\partial x}, \frac{dz}{dx}$.

Solution. We have $\frac{\partial z}{\partial x} = \frac{2x}{x^2 - y^2}$. Making use of the formula for the total derivative, we find

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} = \frac{2x}{x^2 - y^2} - \frac{2ye^x}{x^2 - y^2} = \frac{2(x - ye^x)}{x^2 - y^2}.$$

1235. $z = \frac{1}{2}\ln\frac{u}{v}$, where $u = \tan^2 x$, $v = \cot^2 x$. Find $\frac{dz}{dx}$.

1236. $z = \frac{x^2 - y}{x^2 + y}$, where $y = 3x + 1$. Find $\frac{dz}{dx}$.

1237. $z = x^2 y$, where $y = \cos x$. Find $\frac{\partial z}{\partial x}$ and $\frac{dz}{dx}$.

1238. $z = \ln\frac{x - \sqrt{x^2 - y^2}}{x + \sqrt{x^2 - y^2}}$, where $y = x\cos\alpha$. Find $\frac{dz}{dx}$.

1239. $z = x^2 + y^2$, where $x = \xi + \eta$, $y = \xi - \eta$. Find $\frac{\partial z}{\partial \xi}, \frac{\partial z}{\partial \eta}$.

1240. $u = \ln(x^2 + y^2)$, where $x = \xi\eta$, $y = \frac{\xi}{\eta}$. Find $\frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}$.

1241. Show that the function $u = \ln(1/r)$, where $r = \sqrt{(x - a)^2 + (y - b)^2}$, satisfies the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

8.2.5. Directional derivative. Gradient of a function. The derivative of the function $z = f(x; y)$ at the point $M(x; y)$ *in the direction* of the vector $\mathbf{l} = \overline{MM_1}$ is the limit

$$\frac{\partial z}{\partial l} = \lim_{|MM_1| \to 0} \frac{f(M_1) - f(M)}{|MM_1|} = \lim_{\rho \to 0} \frac{\Delta z}{\rho},$$

where $\rho = \sqrt{\Delta x^2 + \Delta y^2}$.

If the function $f(x, y)$ is differentiable, then the directional derivative can be found from the formula

$$\frac{\partial z}{\partial l} = \frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\sin\alpha,$$

where α is the angle formed by the vector **l** and the x-axis.

In the case of a function of three variables $u = f(x, y, z)$ a similar procedure is used to determine the directional derivative. The corresponding formula has the form

$$\frac{\partial u}{\partial l} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma,$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the vector **l**.

The *gradient* of the function $z = f(x, y)$ at the point $M(x; y)$ is a vector emanating from the point M and possessing the partial derivatives of the function z as its coordinates:

$$\text{grad}\, z = \frac{\partial z}{\partial x}\mathbf{i} + \frac{\partial z}{\partial y}\mathbf{j}.$$

The gradient of the function and the derivative in the direction of the vector **l** are related as

$$\frac{\partial z}{\partial t} = \text{der grad}\, z.$$

The gradient shows the direction of the fastest growth of the function at a given point. The derivative $\frac{\partial z}{\partial l}$ in the direction of the gradient has the greatest value equal to

$$\left(\frac{\partial z}{\partial l}\right)_{\text{gr}} = |\text{grad}\, z| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}.$$

In the case of the function $u = f(x, y, z)$, its gradient is equal to

$$\text{grad}\, u = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}.$$

1242. Find the derivative of the function $z = x^2 - y^2$ at the point $M(1; 1)$ in the direction of the vector **l** making the angle $\alpha = 60°$ with the positive direction of the x-axis.

Solution. Let us find the values of the partial derivative at the point M:

$$\frac{\partial z}{\partial x} = 2x, \frac{\partial z}{\partial y} = -2y, \quad \left(\frac{\partial z}{\partial x}\right)_M = 2, \quad \left(\frac{\partial z}{\partial y}\right)_M = -2.$$

Since $\cos\alpha = \cos 60° = 1/2$, $\sin\alpha = \sin 60° = \sqrt{3}/2$, we have

$$\frac{\partial z}{\partial l} = 2\cdot\frac{1}{2} - \frac{\sqrt{3}}{2} = 1 - \sqrt{3} = -0.7.$$

1243. Find the derivative of the function $u = xy^2z^3$ at the point $M(3; 2; 1)$ in the direction of the vector $\overline{MN}$, where $N(5; 4; 2)$.

Solution. We find the vector $\overline{MN}$ and its direction cosines: $\overline{MN} = \mathbf{l} = (5 - 3)\mathbf{i} + (4 - 2)\mathbf{j} + (2 - 1)\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$;

$$\cos\alpha = \frac{2}{\sqrt{2^2 + 2^2 + 1^2}} = \frac{2}{3}; \quad \cos\beta = \frac{2}{3}, \ \cos\gamma = \frac{1}{3}.$$

Next we calculate the values of the partial derivatives at the point M:

$$\frac{\partial u}{\partial x} = y^2z^3; \ \frac{\partial u}{\partial y} = 2xyz^3; \quad \frac{\partial u}{\partial z} = 3xy^2z^2;$$

$$\left(\frac{\partial u}{\partial x}\right)_M = 4; \quad \left(\frac{\partial u}{\partial y}\right)_M = 12; \quad \left(\frac{\partial u}{\partial z}\right)_M = 36.$$

Consequently,

$$\frac{\partial u}{\partial l} = 4 \cdot \frac{2}{3} + 12 \cdot \frac{2}{3} + 36 \cdot \frac{1}{3} = 22\frac{2}{3}.$$

1244. Find the derivative of the function $z = \ln(x^2 + y^2)$ at the point $M(3; 4)$ in the direction of the gradient of the function z.

Solution. Here the vector $\mathbf{l}$ coincides with the gradient of the function $z = \ln(x^2 + y^2)$ at the point $M(3; 4)$ and is equal to

$$\operatorname{grad} z = \left(\frac{2x}{x^2 + y^2}\right)_M \mathbf{i} + \left(\frac{2y}{x^2 + y^2}\right)_M \mathbf{j} = \frac{6}{25}\mathbf{i} + \frac{8}{25}\mathbf{j}.$$

Consequently,

$$\frac{\partial z}{\partial l} = |\operatorname{grad} z| = \sqrt{\left(\frac{6}{25}\right)^2 + \left(\frac{8}{25}\right)^2} = \frac{2}{5}.$$

1245. Find the value and the direction of the gradient of the function $u = \tan x - x + 3\sin y - \sin^3 y + z + \cot z$ at the point $M(\pi/4; \pi/3; \pi/2)$.

Solution. Let us find the partial derivatives

$$\frac{\partial u}{\partial x} = \sec^2 x - 1, \quad \frac{\partial u}{\partial y} = 3\cos y - 3\sin^2 y \cos y, \quad \frac{\partial u}{\partial z} = 1 - \operatorname{cosec}^2 z$$

and calculate their values at the point $M(\pi/4; \pi/3; \pi/2)$:

$$\left(\frac{\partial u}{\partial x}\right)_M = 2 - 1 = 1, \quad \left(\frac{\partial u}{\partial y}\right)_M = 3 \cdot \frac{1}{2} - 3\left(\frac{\sqrt{3}}{2}\right)^2 \cdot \frac{1}{2} = \frac{3}{8},$$

$$\left(\frac{\partial u}{\partial z}\right)_M = 1 - 1 = 0.$$

It follows that

$$(\operatorname{grad} \nu)_M = \mathbf{i} + \frac{3}{8}\mathbf{j}; \ |\operatorname{grad} u|_M = \sqrt{1^2 + (3/8)^2} = \sqrt{73}/8;$$

$$\cos\alpha = \frac{1}{\sqrt{73/8}} = \frac{8}{\sqrt{73}};\ \cos\beta = \sin\alpha = \frac{3}{\sqrt{73}}.$$

1246. Find the derivative of the function $z = x^2 - xy + y^2$ at the point $M(1; 1)$ in the direction of the vector $\mathbf{l} = 6\mathbf{i} + 8\mathbf{j}$.

1247. Find the derivative of the function $u = \arcsin(z/\sqrt{x^2 + y^2})$ at the point $M(1; 1; 1)$ in the direction of the vector $\overline{MN}$, where $N(3; 2; 3)$.

1248. Find the derivative of the function $u = \ln(x^2 + y^2 + z^2)$ at the point $M(1; 2; 1)$ in the direction of the vector $\mathbf{r} = 2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$.

1249. Find the value and the direction of the gradient of the function $u = 1/r$, where $r = \sqrt{x^2 + y^2 + z^2}$, at the point $M(x_0; y_0; z_0)$.

1250. Find the value and the direction of the gradient of the function $u = xyz$ at the point $M(2; 1; 1)$.

1251. Find the derivative of the function $u = x/2 + y/3 + z/6$ in the direction of $\mathbf{l} = 6\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$ at an arbitrary point.

8.2.6. Differentiation of implicit functions. The derivative of the implicit function $y = y(x)$ specified by the equation $F(x, y) = 0$, where $F(x, y)$ is a differentiable function of the variables x and y, can be calculated by the formula

$$y' = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}}$$

under the condition that $\dfrac{\partial F}{\partial y} \neq 0$.

The higher-order derivatives of the implicit function can be found by a successive differentiation of the indicated formula with y being considered as a function of x.

Similarly, the partial derivatives of the implicit function of two variables $z = \varphi(x, y)$ specified by the equation $F(x, y, z) = 0$, where $F(x, y, z)$ is a differentiable function of the variables x, y, and z, can be calculated by the formulas

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}},\quad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}$$

under the condition that $\dfrac{\partial F}{\partial z} \neq 0$.

1252. $\cos(x + y) + y = 0$. Find y'.

Solution. Here $F(x, y) = \cos(x + y) + y$. We find $\dfrac{\partial F}{\partial x} = -\sin(x + y)$, $\dfrac{\partial F}{\partial y} =$

$= -\sin(x+y) + 1$. Consequently,

$$y' = -\frac{-\sin(x+y)}{1-\sin(x+y)} = \frac{\sin(x+y)}{1-\sin(x+y)}.$$

1253. $y - \sin y = x$. Find y' and y''.

Solution. Here $F(x, y) = y - \sin y - x$. We have $\frac{\partial F}{\partial x} = -1, \frac{\partial F}{\partial u} = 1 - \cos y =$ $= 2\sin^2 \frac{y}{2}$, whence

$$y' = -\frac{-1}{2\sin^2(y/2)} = \frac{1}{2}\operatorname{cosec}^2\frac{y}{2}.$$

Let us find the second derivative:

$$y'' = \frac{1}{2}\cdot 2\operatorname{cosec}\frac{y}{2}\left(-\operatorname{cosec}\frac{y}{2}\cdot\cot\frac{y}{2}\right)\cdot\frac{1}{2}y' = -\frac{1}{4}\operatorname{cosec}^4\frac{y}{2}\cdot\cot\frac{y}{2}.$$

1254. $z^3 - 3xyz = a^3$. Find $\frac{\partial F}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution. Here $F(x, y, z) = z^3 - 3xyz - a^3$. We find $\frac{\partial F}{\partial x} = -3yz, \frac{\partial F}{\partial y} = -3xz,$ $\frac{\partial F}{\partial z} = 3z^2 - 3xy$. Then we have

$$\frac{\partial z}{\partial x} = -\frac{-3yz}{3z^2-3xy} = \frac{yz}{z^2-xy};\quad \frac{\partial z}{\partial y} = -\frac{-3xz}{3z^2-3xy} = \frac{xz}{z^2-xy}.$$

1255. $xyz = x + y + z$. Find dz.

Solution. As is known, $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$. Therefore, we shall first find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$:

$$\frac{\partial z}{\partial x} = -\frac{yz-1}{xy-1},\quad \frac{\partial z}{\partial y} = -\frac{xz-1}{xy-1}.$$

Hence we have

$$dz = -\frac{1}{xy-1}[(yz-1)dx + (xz-1)dy].$$

1256. $x^2 + y^2 + \ln(x^2+y^2) = a^2$. Find y'.
1257. $(y/x) + \sin(y/x) = a$. Find y'.
1258. $(xy-\alpha)^2 + (xy-\beta)^2 = r^2$. Find y', y''.
1259. $x^3 + 2y^3 - 2xy\sqrt{2xy} + 1 = 0$. Find y'.
1260. $\ln\tan(y/x) - y/x = a$. Find y'.
1261. $(x^2+y^2-bx)^2 = a^2(x^2+y^2)$. Find y' at the point $M(b; b)$.
1262. $3\sin(\sqrt{x}/y) - 2\cos(\sqrt{x}/y) + 1 = 0$. Find y'.

1263. $0.5 \ln (x^2 + y^2) - \arctan (y/x) = 0$. Find y'.
1264. $x^2 - x \cdot 2^{y+1} + 4^y - x + 2^y + 2 = 0$. Find y'.
1265. $x + y - e^{x+y} = 0$. Find y', y''.
1266. $x + y + z = e^z$. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.
1267. $x^3 + y^3 + z^3 - 3xyz = 0$. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.
1268. $x = z \ln (z/y)$. Find dz.
1269. $x \sin y + y \sin x + z \sin x = a$. Find $\frac{\partial z}{\partial y}$.
1270. $xy + xz + yz = 1$. Find dz.
1271. $xe^y + ye^x + ze^x = a$. Find $\frac{\partial z}{\partial x}$.
1272. $z = x + \arctan \frac{y}{z - x}$. Find $\frac{\partial z}{\partial x}$.

8.3. Tangent Plane and Normal to a Surface

The term *tangent plane* to a surface at a point M is used for a plane containing all the tangents to the curves drawn on the surface through the point M.

A *normal* to a surface is a straight line passing through the point of tangency M at right angles to the tangent plane.

If the surface is specified by the equation $F(x, y, z) = 0$, then the equation of the tangent plane at the point $M(x_0; y_0; z_0)$ of the surface has the form

$$\left(\frac{\partial F}{\partial x}\right)_M (x - x_0) + \left(\frac{\partial F}{\partial y}\right)_M (y - y_0) + \left(\frac{\partial F}{\partial z}\right)_M (z - z_0) = 0,$$

where $\left(\frac{\partial F}{\partial x}\right)_M, \left(\frac{\partial F}{\partial y}\right)_M, \left(\frac{\partial F}{\partial z}\right)_M$ are the values of the partial derivatives at the point M, and x, y, z are the current coordinates of the point of the tangent plane.

The equations of the normal to the surface at the point M are written as

$$\frac{x - x_0}{\left(\frac{\partial F}{\partial x}\right)_M} = \frac{y - y_0}{\left(\frac{\partial F}{\partial y}\right)_M} = \frac{z - z_0}{\left(\frac{\partial F}{\partial z}\right)_M}.$$

Here x, y, z are the current coordinates of the point of the normal.

Now if the equation of the surface is given in an explicit form $z = f(x, y)$, then the equation of the tangent plane at the point $M(x_0; y_0; z_0)$ is written in the form

$$z - z_0 = \left(\frac{\partial z}{\partial x}\right)_M (x - x_0) + \left(\frac{\partial z}{\partial y}\right)_M (y - y_0),$$

and the equations of the normal in the form

$$\frac{x - x_0}{\left(\frac{\partial z}{\partial x}\right)_M} = \frac{y - y_0}{\left(\frac{\partial z}{\partial y}\right)_M} = \frac{z - z_0}{-1}.$$

1273. Given the surface $z = x^2 - 2xy + y^2 - x + 2y$. Set up the equation of the tangent plane and the equations of the normal to the surface at the point $M(1; 1; 1)$.

Solution. Let us find the partial derivatives $\frac{\partial z}{\partial x} = 2x - 2y - 1$ and $\frac{\partial z}{\partial y} = -2x + 2y + 2$ and their values at the point $M(1; 1; 1)$: $\left(\frac{\partial z}{\partial x}\right)_M = -1$, $\left(\frac{\partial z}{\partial y}\right)_M = 2$.

The equation of the tangent plane is

$$z - 1 = -(x - 1) + 2(y - 1), \quad \text{or} \quad x - 2y + z = 0.$$

The equations of the normal are

$$(x - 1)/(-1) = (y - 1)/2 = (z - 1)/(-1).$$

1274. Draw the tangent planes, parallel to the plane $x + y + z = 1$, to the surface $x^2 + 2y^2 + 3z^2 = 11$.

Solution. Here $F(x, y, z) = x^2 + y^2 + 3z^2 - 11$. We find the partial derivatives:

$$\frac{\partial F}{\partial x} = 2x, \quad \frac{\partial F}{\partial y} = 4y, \quad \frac{\partial F}{\partial z} = 6z.$$

If follows from the condition of parallelism of the tangent plane to the given plane that $(\partial F/\partial x)/1 = (\partial F/\partial y)/1 = (\partial F/\partial z)/1$, or $(2x)/1 = (4y)/1 = (6z)/1$. Adding to these equations the equation of the surface $x^2 + 2y^2 + 3z^2 = 11$, we find the coordinates of the points of tangency: $M_1(\sqrt{6}; \sqrt{6}/2; \sqrt{6}/3)$ and $M_2(-\sqrt{6}; -\sqrt{6}/2; -\sqrt{6}/3)$. Consequently, the equations of the tangent planes have the form

$$1 \cdot (x \pm \sqrt{6}) + 1 \cdot (y \pm \sqrt{6}/2) + 1 \cdot (z \pm \sqrt{6}/3) = 0,$$

that is,

$$x + y + z + 11/\sqrt{6} = 0 \quad \text{and} \quad x + y + z - 11/\sqrt{6} = 0.$$

1275. Find the equations of the tangent plane and the normal to the surface $z = 1 + x^2 + y^2$ at the point $M(1; 1; 3)$.

1276. Find the equations of the tangent plane and the normal to the surface $x^2 + y^2 - z^2 = -1$ at the point $M(2; 2; 3)$.

1277. Find the equations of the tangent plane and the normal to the surface $z = \ln(x^2 + y^2)$ at the point $M(1; 0; 0)$.

1278. Find the equations of the tangent plane and the normal to the surface $z = \sin x \cos y$ at the point $M(\pi/4; \pi/4; 1/2)$.

1279. Derive the equations of the tangent planes to the surface $x^2 + 2y^2 + 3z^2 = 21$ which are parallel to the plane $x + 4y + 6z = 0$.

1280. Prove that the tangent planes to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}$ $(a > 0)$ intercept line segments on the coordinate axes whose sum is constant.

1281. At what point of the ellipsoid $x^2/4 + y^2/4 + z^2 = 1$ does its normal form equal angles with the coordinate axes?

1282. Prove that $\frac{\partial z}{\partial x} = -\frac{\cos\alpha}{\cos\gamma}$, $\frac{\partial z}{\partial y} = -\frac{\cos\beta}{\cos\gamma}$, if $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the normal to the surface $z = f(x, y)$.

8.4. Extremum of a Function of Two Independent Variables

8.4.1. Extremum of a function. The function $z = f(x, y)$ possesses a *maximum* (*minimum*) at the point $M_0(x_0; y_0)$ if its value at that point is greater (smaller) than that at any other point $M(x; y)$ belonging to a certain neighbourhood of the point M_0, i.e. $f(x_0, y_0) > f(x, y)$ ($f(x_0, y_0) < f(x, y)$ respectively) for all the points $M(x, y)$ complying with the condition $|M_0M| < \delta$, where δ is a sufficiently small positive number.

A maximum or a minimum of a function is called its *extremum*. The point M_0 at which the function possesses an extremum is known as the *point of extremum*.

If the differentiable function $z = f(x, y)$ attains its extremum at the point $M_0(x_0; y_0)$, then its first-order partial derivatives at that point are equal to zero, i.e.

$$\frac{\partial f(x_0, y_0)}{\partial x} = 0, \quad \frac{\partial f(x_0, y_0)}{\partial y} = 0$$

(**necessary conditions** for an extremum).

The points at which the partial derivatives are equal to zero are called *stationary points*. Not every stationary point is a point of extremum.

Assume $M_0(x_0, y_0)$ to be a stationary point of the function $z = f(x, y)$. Let us introduce the designations

$$A = \frac{\partial^2 f(x_0, y_0)}{\partial x^2}, \quad B = \frac{\partial^2 f(x_0, y_0)}{\partial x\, \partial y}, \quad C = \frac{\partial^2 f(x_0, y_0)}{\partial y^2}$$

and derive the *discriminant* $\Delta = AC - B^2$. Then:

if $\Delta > 0$, the function possesses an extremum at the point M_0, namely, a maximum for $A < 0$ (or $C < 0$) and a minimum for $A > 0$ (or $C > 0$);

if $\Delta < 0$, then there is no extremum at the point M_0 (**sufficient conditions** for the existence or the absence of the extremum);

if $\Delta = 0$, further investigation is required (doubtful case).

1283. Find the extremum of the function $z = x^2 + xy + y^2 - 3x - 6y$.

Solution. We find the first-order partial derivatives:

$$\frac{\partial z}{\partial x} = 2x + y - 3, \quad \frac{\partial z}{\partial y} = x + 2y - 6.$$

Making use of the necessary conditions for an extremum, we find the stationary points:

$$\begin{cases} 2x + y - 3 = 0, \\ x + 2y - 6 = 0, \end{cases}$$

whence $x = 0$, $y = 3$; $M(0; 3)$.

Next we find the second-order partial derivatives at the point M:

$$\frac{\partial^2 z}{\partial x^2} = 2, \quad \frac{\partial^2 z}{\partial y^2} = 2, \quad \frac{\partial^2 z}{\partial x\,\partial y} = 1$$

and derive the discriminant:

$$\Delta = AC - B^2 = 2 \cdot 2 - 1 = 3 > 0; \quad A > 0.$$

Consequently, at the point $M(0; 3)$ the given function possesses a minimum. The value of the function at that point is $z_{\min} = -9$.

1284. Find the extremum of the function

$$z = \frac{1}{2}xy + (47 - x - y)\left(\frac{x}{3} + \frac{y}{4}\right).$$

Solution. We find the first-order partial derivatives:

$$\frac{\partial z}{\partial x} = -\frac{1}{12}y - \frac{2}{3}x + \frac{47}{3}, \quad \frac{\partial z}{\partial y} = -\frac{1}{2}y - \frac{1}{12}x + \frac{47}{4}.$$

Making use of the necessary conditions for an extremum, we find the stationary points:

$$\begin{cases} -\dfrac{1}{12}y - \dfrac{2}{3}x + \dfrac{47}{3} = 0, \\ -\dfrac{1}{2}y - \dfrac{1}{12}x + \dfrac{47}{4} = 0, \end{cases} \quad \text{or} \quad \begin{cases} 8x + y = 188, \\ x + 6y = 141. \end{cases}$$

From this we find $x = 21$, $y = 20$; the stationary point $M(21; 20)$.

Now we find the values of the second-order partial derivatives at the point M:

$$\frac{\partial^2 z}{\partial x^2} = -\frac{2}{3}, \quad \frac{\partial^2 z}{\partial y^2} = -\frac{1}{2}, \quad \frac{\partial^2 z}{\partial x\,\partial y} = -\frac{1}{12}.$$

Then we have

$$\Delta = AC - B^2 = \left(-\frac{2}{3}\right)\left(-\frac{1}{2}\right) - \left(-\frac{1}{12}\right)^2 = \frac{1}{3} - \frac{1}{144} > 0.$$

Since $A < 0$, the function possesses a maximum $z_{\max} = 282$ at the point $M(21; 20)$.

Find the extrema of the following functions:

1285. $z = xy^2(1 - x - y)$. **1286.** $z = x^3 + y^3 - 15xy$.
1287. $z = 4 - (x^2 + y^2)^{2/3}$. **1288.** $z = (x^2 + y^2)(e^{-(x^2+y^2)} - 1)$.
1289. $z = \sqrt{(a - x)(a - y)(x + y - a)}$.

8.4.2. Conditional extremum. The greatest and the least value of a function in a closed domain. The term *conditional extremum* of the function $z = f(x, y)$ is used for the extremum of that function attained under the condition that the variables x and y are connected by the equation $\varphi(x, y) = 0$ (*equation of constraint*).

The search for a conditional extremum can be reduced to the test for an ordinary extremum of the so-called *Lagrange's function*

$$u = f(x, y) + \lambda\varphi(x, y),$$

where λ is an indeterminate constant multiplier.

The necessary conditions for an extremum of the Lagrange's function have the form

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial f}{\partial x} + \lambda\dfrac{\partial \varphi}{\partial x} = 0, \\[2ex] \dfrac{\partial u}{\partial y} = \dfrac{\partial f}{\partial y} + \lambda\dfrac{\partial \varphi}{\partial y} = 0, \\[2ex] \varphi(x, y) = 0. \end{cases}$$

This system of three equations yields the unknowns x, y and λ.

To find the *greatest* and the *least* value of the function in a closed domain, it is necessary

(1) to find the stationary points belonging to that domain and calculate the values of the function at those points;

(2) to find the greatest and the least value of the function on the curves forming the boundary of the domain;

(3) to choose the greatest and the least value from among all the values found.

1290. Find the extremum of the function $z = xy$ under the condition that x and y are related as $2x + 3y - 5 = 0$.

Solution. Let us consider the Lagrange function $u = xy + \lambda(2x + 3y - 5)$. We have $\dfrac{\partial u}{\partial x} = y + 2\lambda$, $\dfrac{\partial u}{\partial y} = x + 3\lambda$. From the system of equations (the necessary conditions for an extremum)

$$\begin{cases} y + 2\lambda = 0, \\ x + 3\lambda = 0, \\ 2x + 3y - 5 = 0 \end{cases}$$

we find that $\lambda = -5/12$, $x = 5/4$, $y = 5/6$. It is easy to see that at the point (5/4; 5/6) the function $z = xy$ attains its greatest value $z_{max} = 25/24$.

1291. Among all the right triangles with a given area S find one whose hypotenuse has the least value.

Solution. Assume that x and y are the legs of the triangle and z is the hypotenuse. Since $z^2 = x^2 + y^2$, the problem reduces to finding the least value of function $x^2 + y^2$ under the conditions that x and y are connected by the equation $xy/2 = S$, i.e. $xy - 2S = 0$. Let us consider the function $u = x^2 + y^2 + \lambda(xy - 2S)$ and find its partial derivatives

$$\frac{\partial u}{\partial x} = 2x + \lambda y, \quad \frac{\partial u}{\partial y} = 2y + \lambda x.$$

Since $x > 0$, $y > 0$, the system of equations

$$\begin{cases} 2x + \lambda y = 0, \\ 2y + \lambda x = 0, \\ xy/2 = S \end{cases}$$

yields the solution $\lambda = -2$, $x = y = \sqrt{2S}$.

Thus we see that the hypotenuse has the least value if the legs of the triangle are equal to each other.

1292. Find the least and the greatest value of the function $z = x^2 + y^2$ on the circle $(x - \sqrt{2})^2 + (y - \sqrt{2})^2 \leqslant 9$.

Solution. Here we consider the domain D bounded by the circle $(x - \sqrt{2})^2 + (y - \sqrt{2})^2 = 9$, the points of the circle inclusive.

Let us find the stationary points of the given function; we have $\dfrac{\partial z}{\partial x} = 2x$, $\dfrac{\partial z}{\partial y} = 2y$; by virtue of the necessary conditions for an extremum we find that $x = 0$, $y = 0$.

It is easy to see that at the point (0; 0) the function $z = x^2 + y^2$ possesses the least value $z_{\text{least}} = 0$, the indicated point being an interior point of the domain D.

Next we test the function $z = x^2 + y^2$ for a conditional extremum in the case when x and y are related as $(x - \sqrt{2})^2 + (y - \sqrt{2})^2 = 9$. We consider the function

$$u = x^2 + y^2 + \lambda[(x - \sqrt{2})^2 + (y - \sqrt{2})^2 - 9].$$

Now we find the partial derivatives

$$\frac{\partial u}{\partial x} = 2x + 2\lambda(x - \sqrt{2}), \quad \frac{\partial u}{\partial y} = 2y + 2\lambda(y - \sqrt{2}).$$

To determine x, y and λ, we obtain a system of equations

$$\begin{cases} x + \lambda(x - \sqrt{2}) = 0, \\ y + \lambda(y - \sqrt{2}) = 0, \\ (x - \sqrt{2})^2 + (y - \sqrt{2})^2 = 9. \end{cases}$$

This system has two solutions: $x = y = 5\sqrt{2}/2$, $\lambda = -5/3$ and $z = 25$, $x = y = -\sqrt{2}/2$, $\lambda = -1/3$ and $z = 1$. Hence, the function attains its greatest value at the point $(5\sqrt{2}/2;\ 5\sqrt{2}/2)$.

Thus, we have $z_{least} = 0$, $z_{gr} = 25$.

1293. Find the extremum of the function $z = x^2 + y^2$ if x and y are related as $x/4 + y/3 = 1$.

1294. Find the least and the greatest value of the function $z = x^2 - xy + y^2 - 4x$ in a closed domain bounded by the straight lines $x = 0$, $y = 0$, $2x + 3y - 12 = 0$.

1295. Find the least and the greatest value of the function $z = xy + x + y$ on a square bounded by the straight lines $x = 1$, $x = 2$, $y = 2$, $y = 3$.

1296. Find the least and the greatest value of the function $z = xy$ on a circle $x^2 + y^2 \leqslant 1$.

1297. Find the least and the greatest value of the function $z = x^2 + 3y^2 + x - y$ on a triangle bounded by the straight lines $x = 1$, $y = 1$, $x + y = 1$.

1298. Find the least and the greatest value of the function $z = 1 - x^2 - y^2$ on a circle $(x - 1)^2 + (y - 1)^2 \leqslant 1$.

1299. Find the least and the greatest value of the function $z = \sin x + \sin y + \sin (x + y)$ in the domain $0 \leqslant x \leqslant \pi/2$, $0 \leqslant y \leqslant \pi/2$.

1300. Find the least and the greatest value of the function $z = \sin x + \sin y + \sin (x + y)$ in the domain $0 \leqslant x \leqslant 3\pi/2$, $0 \leqslant y \leqslant 3\pi/2$.

1301. Find the least and the greatest value of the function $z = \cos x \cos y \cos (x + y)$ in the domain $0 \leqslant x \leqslant \pi$, $0 \leqslant y \leqslant \pi$.

1302. Of all the triangles inscribed into a circle choose the one whose area is the greatest.

1303. Of all the triangles with a given perimeter find the one whose area is the greatest.

1304. Of all the rectangles with a given area S find the one whose perimeter has the least value.

1305. Find the dimensions of the rectangular parallelepiped having the maximum volume for the given total surface S.

Chapter 9

Indefinite Integrals

9.1. Direct Integration. Change of Variable and Integration by Parts

9.1.1. Direct integration. The function $F(x)$ is called *a primitive* or an *antiderivative* of the function $f(x)$ if $F'(x) = f(x)$ or $dF(x) = f(x)\,dx$.

If the function $f(x)$ possesses an antiderivative $F(x)$, then it possesses infinitely many antiderivatives, all the antiderivatives being contained in the expression $F(x) + C$, where C is a constant.

An *indefinite integral* of the function $f(x)$ (or of the expression $f(x)\,dx$) is a collection of all its antiderivatives. The notation is

$$\int f(x)dx = F(x) + C.$$

Here $\int$ is the integral sign, $f(x)$ is the integrand, $f(x)\,dx$ is the element of integration, and x is the variable of integration.

The process of finding an indefinite integral is called *integration* of a function.

Properties of an indefinite integral (rules of integration)

1°. $(\int f(x)dx)' = f(x)$.

2°. $d(\int f(x)\,dx) = f(x)\,dx$.

3°. $\int dF(x) = F(x) + C$.

4°. $\int af(x)\,dx = a\int f(x)dx$, where a is a constant.

5°. $\int [f_1(x) \pm f_2(x)]dx = \int f_1(x)\,dx \pm \int f_2(x)\,dx$.

6°. If $\int f(x)\,dx = F(x) + C$ and $u = \varphi(x)$, then $\int f(x)\,du = F(u) + C$.

Table of basic integrals

I. $\int dx = x + C$.

II. $\displaystyle\int x^m dx = \frac{x^{m+1}}{m+1} + C$ for $m \neq -1$.

III. $\displaystyle\int \frac{dx}{x} = \ln|x| + C$.

IV. $\displaystyle\int \frac{dx}{1+x^2} = \arctan x + C$.

V. $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$.

VI. $\int e^x dx = e^x + C.$

VII. $\int a^x dx = \dfrac{a^x}{\ln a} + C.$

VIII. $\int \sin x\, dx = -\cos x + C.$

IX. $\int \cos x\, dx = \sin x + C.$

X. $\int \sec^2 x\, dx = \tan x + C.$

XI. $\int \operatorname{cosec}^2 x\, dx = -\cot x + C.$

XII. $\int \sinh x\, dx = \cosh x + C.$

XIII. $\int \cosh x\, dx = \sinh x + C.$

XIV. $\int \dfrac{dx}{\cosh^2 x} = \tanh x + C.$

XV. $\int \dfrac{dx}{\sinh^2 x} = -\coth x + C.$

1306. Find the integral $\int (2x^3 - 5x^2 + 7x - 3)\, dx.$

Solution. Using properties 4° and 5°, we get

$$\int (2x^3 - 5x^2 + 7x - 3)\, dx = 2\int x^3\, dx - 5\int x^2\, dx + 7\int x\, dx - 3\int dx.$$

We apply formula II to the first three integrals on the right-hand side and formula I to the fourth integral:

$$\int (2x^3 - 5x^2 + 7x - 3)\, dx = 2 \cdot \frac{x^4}{4} - 5 \cdot \frac{x^3}{3} + 7 \cdot \frac{x^2}{2} - 3x + C$$
$$= \frac{1}{2}x^4 - \frac{5}{3}x^3 + \frac{7}{2}x^2 - 3x + C.$$

1307. Find the integral $\int \left(\sqrt{x} + \dfrac{1}{\sqrt[3]{x}}\right)^2 \cdot dx.$

Solution. We have

$$\int \left(\sqrt{x} + \frac{1}{\sqrt[3]{x}}\right)^2 dx = \int \left(x + 2 \cdot \frac{x^{1/2}}{x^{1/3}} + \frac{1}{x^{2/3}}\right) dx$$
$$= \int (x + 2x^{1/6} + x^{-2/3})\, dx = \int x\, dx + 2\int x^{1/6}\, dx + \int x^{-2/3}\, dx$$
$$= \frac{x^2}{2} + 2 \cdot \frac{x^{7/6}}{7/6} + \frac{x^{1/3}}{1/3} + C = \frac{x^2}{2} + \frac{12}{7}x\sqrt[6]{x} + 3\sqrt[3]{x} + C.$$

1308. Find the integral $\int 2^x \cdot 3^{2x} \cdot 5^{3x} dx.$

Solution. We have

$$\int 2^x \cdot 3^{2x} \cdot 5^{3x} dx = \int (2 \cdot 3^2 \cdot 5^3)^x dx = \int 2250^x dx = \frac{2250^x}{\ln 2250} + C.$$

Property 6° makes it possible to expand considerably the table of basic integrals by means of *inserting the function under the differential sign*.

1309. Find the integral $\int (1 + x^2)^{1/2} x\, dx$.

Solution. This integral can be reduced to formula II by transforming it as follows:

$$\int (1 + x^2)^{1/2} x\, dx = \frac{1}{2}\int (1 + x^2)^{1/2} \cdot 2x\, dx = \frac{1}{2}\int (1 + x^2)^{1/2} d(1 + x^2).$$

We have got the expression $1 + x^2$ as the variable of integration and obtained an integral of a power function with respect to this variable. Consequently,

$$\int (1 + x^2)^{1/2} x\, dx = \frac{1}{2} \cdot \frac{(1 + x^2)^{1/2+1}}{1/2 + 1} + C = \frac{1}{3}(1 + x^2)^{3/2} + C.$$

1310. Find the integral $\int (x^2 - 3x + 1)^{10} \cdot (2x - 3) dx$.

Solution. Proceeding in the same way as in the previous example, we obtain

$$\int (x^2 - 3x + 1)^{10}\, d(x^2 - 3x + 1) = \frac{1}{11}(x^2 - 3x + 1)^{11} + C.$$

1311. Find the integral $\int (\ln t)^4 \frac{dt}{t}$.

Solution. The expression dt/t can be written as $d(\ln t)$, and therefore

$$\int (\ln t)^4 d(\ln t) = \frac{1}{5}(\ln t)^5 + C.$$

1312. Find the integral $\int e^{3\cos x} \sin x\, dx$.

Solution. The given integral can be represented as follows:

$$\int e^{3\cos x} \sin x dx = \frac{1}{3}\int e^{3\cos x} \cdot 3 \sin x\, dx,$$

but $3 \sin x\, dx = -d(3 \cos x)$, and therefore

$$\int e^{3\cos x} \cdot \sin x\, dx = -\frac{1}{3}\int e^{3\cos x} \cdot d(3 \cos x),$$

that is, the variable of integration is $3 \cos x$. Consequently, the integral is taken by formula VI:

$$\int e^{3\cos x} \sin x\, dx = -\frac{1}{3} e^{3\cos x} + C.$$

1313. Find the integral $\int (2 \sin x + 3 \cos x)\, dx$.

Solution. We find

$$\int (2 \sin x + 3 \cos x)\, dx = 2\int \sin x\, dx + 3\int \cos x\, dx = -2 \cos x + 3 \sin x + C$$

(see formulas VIII and IX).

1314. Find the integral $\int (\tan x + \cot x)^2 dx$.

Solution. We have

$$\int (\tan x + \cot x)^2 dx = \int (\tan^2 x + 2\cot x \cdot \tan x + \cot^2 x)dx$$
$$= \int (\tan^2 x + 1 + 1 + \cot^2 x)dx = \int (\tan^2 x + 1)dx$$
$$+ \int (1 + \cot^2 x)dx = \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx$$
$$= \tan x - \cot x + C$$

(see formulas X and XI).

Find the following integrals:

1315. $\int x\sqrt{x}dx$. **1316.** $\int \frac{dx}{\sqrt[3]{x}}$. **1317.** $\int \frac{2 - \sqrt{1 - x^2}}{\sqrt{1 - x^2}} dx$.

1318. $\int \frac{2 - x^4}{1 + x^2} dx$. **1319.** $\int e^{3x} \cdot 3^x dx$. **1320.** $\int \tan^2 x dx$.

1321. $\int (\sinh x - \sin x)dx$. **1322.** $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$.

1323. $\int (2\tan x + 3\cot x)^2 dx$. **1324.** $\int x\cos(x^2)dx$.

1325. $\int \frac{dx}{x \ln x}$. **1326.** $\int (ax^2 + b)^{1/3} \cdot x dx$.

1327. $\int \sqrt{\sin x} \cos x\, dx$. **1328.** $\int \sin(a + bx)dx$.

1329. $\int \cos(\sin x) \cdot \cos x\, dx$.

9.1.2. Change of variable in an indefinite integral. A *change of variable in an indefinite integral* is performed by means of *substitutions* of two types:

(1) $x = \varphi(t)$, where $\varphi(t)$ is a monotonic, continuously differentiable function of the new variable t. In this case, the integration is carried out by the formula

$$\int f(x)dx = \int f[\varphi(t)]\varphi'(t)dt;$$

(2) $u = \psi(x)$, where u is a new variable. With the use of this substitution, a change of variable is performed by the formula

$$\int f[\psi(x)]\psi'(x)dx = \int f(u)du.$$

1330. Find the integral $\int \frac{\sin\sqrt[3]{x}}{\sqrt[3]{x^2}} dx$.

Solution. Let us make the substitution $t = \sqrt[3]{x}$, i.e. $x = t^3$. As a result of this substitution the variable of integration, rather than its root, turns out to be under the sine sign. We find the differential $dx = 3t^2dt$. This yields

$$\int \frac{\sin\sqrt[3]{x}}{\sqrt[3]{x^2}} dx = \int \frac{3t^2 \sin t}{t^2} dt = 3\int \sin t\, dt = -3\cos t + C.$$

The answer must be expressed in terms of the old variable x. Substituting $t = \sqrt[3]{x}$ into the result of integration, we get

$$\int \frac{\sin\sqrt[3]{x}}{\sqrt[3]{x^2}}\,dx = -3\cos\sqrt[3]{x} + C.$$

1331. Find the integral $\int(2x + 1)^{20}dx$.

Solution. This integral can be found without a change of variable. In this case, it is sufficient to expand the expression $(2x + 1)^{20}$ by Newton's binomial formula and perform a term-by-term integration. However, this technique involves laborious calculations. By means of a change of variable we can easily reduce the integral to a tabular one.

Putting $2x + 1 = t$, we get $2dx = dt$, i.e. $dx = (1/2)dt$, which yields

$$\int (2x + 1)^{20}dx = \frac{1}{2}\int t^{20}dt = \frac{1}{2}\cdot\frac{1}{21}t^{21} + C = \frac{1}{42}(2x + 1)^{21} + C.$$

In general, if the integral $\int f(x)dx$ is given in the table, then it is easy to find the integral $\int f(ax + b)dx$ by means of the substitution $ax + b = t$.

For instance, let us apply this substitution to the integral $\int\sin(ax + b)dx$. We get $ax + b = t$, $a\,dx = dt$ and $dx = (1/a)dt$. Consequently,

$$\int \sin(ax + b)dx = \int \sin t\cdot\frac{dt}{a} = \frac{1}{a}\int \sin t\,dt = -\frac{1}{a}\cos t + C.$$

Returning to the old variable, we get

$$\int \sin(ax + b)dx = -\frac{1}{a}\cos(ax + b) + C.$$

By analogy, it can be shown that

$$\int \cos(ax + b)dx = \frac{1}{a}\sin(ax + b) + C,\quad \int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + C, \text{ etc.}$$

When calculating the integral $\int f(ax + b)dx$, we can, in fact, do without writing down the substitution $ax + b = t$ itself. Here it is sufficient to take into account that $dx = \frac{1}{a}d(ax + b)$. Thus we have

$$\int f(ax + b)dx = \frac{1}{a}F(ax + b) + C,$$

where F is an antiderivative of f.

1332. Find the integral $\int x^2\sqrt{x^3 + 5}\,dx$.

Solution. Putting $\sqrt{x^3 + 5} = t$, we get $x^3 + 5 = t^2$. Next we differentiate both sides of the equation: $3x^2dx = 2t\cdot dt$. Hence, $x^2dx = (2/3)t\,dt$. Consequently,

$$\int x^2\sqrt{x^3 + 5}\,dx = \int \sqrt{x^3 + 5}\,x^2dx = \int t\cdot\frac{2}{3}t\,dt$$

$$= \frac{2}{3}\int t^2 dt = \frac{2}{9}t^3 + C = \frac{2}{9}(\sqrt{x^3+5})^3 + C$$
$$= \frac{2}{9}(x^3+5)\sqrt{x^3+5} + C.$$

The given integral can also be found by means of the substitution $x^3 + 5 = t$. This substitution immediately reduces the integral to a tabular one because the first factor x^2 of the integrand differs from the derivative of the radicand $x^3 + 5$ only by the factor 1/3, i.e. $x^2 = (1/3)(x^3 + 5)'$.

In general, if the integrand is a product of two factors, one of which depends on some function $\psi(x)$ and the other is the derivative of $\psi(x)$ (with an accuracy to within a constant factor), then it is expedient to make a change of variable by the formula $\psi(x) = t$.

1333. Find the integral $\int \frac{(2\ln x + 3)^3}{x}dx$.

Solution. Let us rewrite the given integral in the form $(2\ln x + 3)^3 \frac{1}{x}dx$. Since the derivative of the expression $2\ln x + 3$ is equal to $2/x$ and the second multiplier $1/x$ differs from this derivative only by the constant coefficient 2, we must make the substitution $2\ln x + 3 = t$. Then $2 \cdot \frac{dx}{x} = dt$, $\frac{dx}{x} = \frac{1}{2}dt$. Consequently,

$$\int (2\ln x + 3)^3 \cdot \frac{dx}{x} = \int t^3 \cdot \frac{1}{2}dt = \frac{1}{2}\int t^3 dt = \frac{1}{8}t^4 + C = \frac{1}{8}(2\ln x + 3)^4 + C.$$

1334. Find the integral $\int \frac{f'(x)}{f(x)}dx$.

Solution. Let us carry out the substitution $f(x) = t$. Then $f'(x)dx = dt$ and

$$\int \frac{f'(x)}{f(x)}dx = \int \frac{dt}{t} = \ln|t| + C = \ln|f(x)| + C.$$

For example,

$$\int \frac{xdx}{x^2+1} = \frac{1}{2}\int \frac{2xdx}{x^2+1} = \frac{1}{2}\ln(x^2+1) + C.$$

We do not write the modulus sign here since $x^2 + 1 > 0$.

1335. Find the integral $\int \frac{f'(x)dx}{\sqrt{f(x)}}$.

Solution. Let us put $f(x) = t$. Then we have $f'(x)dx = dt$ and

$$\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}dt = \frac{t^{1/2}}{1/2} + C = 2\sqrt{t} + C = 2\sqrt{f(x)} + C.$$

Note that the integral could have been found by means of the substitution $\sqrt{f(x)} = t$.

1336. Find the integral $\int \frac{dx}{x^2 + a^2}$, if $a \neq 0$.

Solution. To reduce the integral to a tabular one (see formula IV), we divide the numerator and the denominator of the element of integration by a^2:

$$\int \frac{dx}{x^2 + a^2} = \int \frac{(dx)/a^2}{1 + (x/a)^2} = \frac{1}{a}\int \frac{d(x/a)}{1 + (x/a)^2}.$$

We have put the constant factor $1/a$ under the differentiation sign. Considering x/a as a new variable, we get

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\frac{x}{a} + C.$$

The substitution $x = at$ would lead to the same result.

1337. Find the integral $\int \frac{dx}{\sqrt{a^2 - x^2}}$, if $a > 0$.

Solution. Dividing the numerator and the denominator by a, we get

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{(dx)/a}{\sqrt{1 - (x/a)^2}} = \int \frac{d(x/a)}{\sqrt{1 - (x/a)^2}}.$$

Assuming x/a to be a new variable, we obtain

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C.$$

Let us now supplement the table of basic integrals with the following formulas:

XVI. $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C.$

XVII. $\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + C.$

XVIII. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\frac{x}{a} + C.$

XIX. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x - a}{x + a}\right| + C.$

XX. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C.$

XXI. $\displaystyle\int \frac{dx}{\sqrt{x^2 + \lambda}} = \ln|x + \sqrt{x^2 + \lambda}| + C.$

XXII. $\displaystyle\int \frac{dx}{\sin x} = \ln\left|\tan\frac{x}{2}\right| + C = \ln|\operatorname{cosec}x - \cot x| + C.$

XXIII. $\displaystyle\int \frac{dx}{\cos x} = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| = \ln|\sec x + \tan x| + C.$

XXIV. $\int \tan x dx = -\ln|\cos x| + C.$

XXV. $\int \cot x dx = \ln|\sin x| + C.$

It is advisable to memorize formulas I-XXV since most of the integrals used in practical calculations reduce to the integrals which can be computed by these formulas.

1338. Find the integral $\displaystyle\int \frac{dx}{x\sqrt{2x - 9}}$.

Solution. We make the substitution $\sqrt{2x - 9} = t$; then $2x - 9 = t^2$, $x = (t^2 + 9)/2$ and $dx = tdt$. Thus, it follows that

$$\int \frac{dx}{x\sqrt{2x - 9}} = \int \frac{t\,dt}{\frac{t^2 + 9}{2} \cdot t} = 2 \cdot \int \frac{dt}{t^2 + 9}.$$

Applying formula XVIII, we get

$$\int \frac{dx}{x\sqrt{2x - 9}} = \frac{2}{3}\arctan\frac{t}{3} + C = \frac{2}{3}\arctan\frac{\sqrt{2x - 9}}{3} + C.$$

1339. Find the integral $\displaystyle\int \frac{\sin 2x dx}{\sqrt{3 - \cos^4 x}}$.

Solution. We make the substitution $\cos^2 x = t$; then $-2\cos x \sin x dx = dt$, i.e. $\sin 2x dx = -dt$. Now we find

$$\int \frac{\sin 2x}{\sqrt{3 - \cos^4 x}}dx = -\int \frac{dt}{\sqrt{3 - t^2}} = -\arcsin\frac{t}{\sqrt{3}} + C = -\arcsin\frac{\cos^2 x}{\sqrt{3}} + C.$$

(we have used formula XX).

1340. Find the integral $\displaystyle\int \left(2\sin\frac{x}{2} + 3\right)^2 \cos\frac{x}{2}\,dx$.

Solution. We make the substitution $2\sin(x/2) + 3 = t$; then $\cos(x/2)dx = dt$ and

$$\int \left(2\sin\frac{x}{2} + 3\right)^2 \cos\frac{x}{2}\,dx = \int t^2 dt = \frac{1}{3}t^3 + C = \frac{1}{3}\left(2\sin\frac{x}{2} + 3\right)^3 + C.$$

1341. Find the integral $\int \frac{x^4 dx}{\sqrt{x^{10} - 2}}$.

Solution. We make the substitution $x^5 = t$; then $5x^4 dx = dt, x^4 dx = (1/5)dt$ and

$$\int \frac{x^4 dx}{\sqrt{x^{10} - 2}} = \frac{1}{5} \int \frac{dt}{\sqrt{t^2 - 2}} = \frac{1}{5} \ln |t + \sqrt{t^2 - 2}| + C$$

(see formula XXI). Thus, we have

$$\int \frac{x^4 dx}{\sqrt{x^{10} - 2}} = \frac{1}{5} \ln |x^5 + \sqrt{x^{10} - 2}| + C.$$

1342. Find the integral $\int \frac{x dx}{x^4 + 2x^2 + 5}$.

Solution. Transforming the denominator of the fraction, we get $x^4 + 2x^2 + 5 = (x^2 + 1)^2 + 4$. Making the substitution $x^2 + 1 = t$, we obtain $x dx = (1/2)dt$. Hence,

$$\int \frac{x dx}{x^4 + 2x^2 + 5} = \frac{1}{2} \int \frac{dt}{t^2 + 4} = \frac{1}{2} \cdot \frac{1}{2} \arctan \frac{t}{2} + C$$

(see formula XVIII). Thus, it follows that

$$\int \frac{x dx}{x^4 + 2x^2 + 5} = \frac{1}{4} \arctan \frac{x^2 + 1}{2} + C.$$

1343. Find the integral $\int \frac{e^{2x}}{e^{4x} - 5} dx$.

Solution. We put $e^{2x} = t$; then $e^{2x} dx = (1/2)dt$ and

$$\int \frac{e^{2x} dx}{e^{4x} - 5} = \frac{1}{2} \int \frac{dt}{t^2 - 5} = \frac{1}{2} \cdot \frac{1}{2\sqrt{5}} \ln \left| \frac{t - \sqrt{5}}{t + \sqrt{5}} \right| + C$$

(we have applied formula XIX).

Thus,

$$\int \frac{e^{2x} dx}{e^{4x} - 5} = \frac{1}{4\sqrt{5}} \ln \left| \frac{e^{2x} - \sqrt{5}}{e^{2x} + \sqrt{5}} \right| + C.$$

1344. Find the integral $\int \frac{e^{2x} dx}{e^{4x} + 5}$.

Solution. Performing the same substitution as in the previous example, we obtain

$$\int \frac{e^{2x} dx}{e^{4x} + 5} = \frac{1}{2} \int \frac{dt}{t^2 + 5} = \frac{1}{2\sqrt{5}} \arctan \frac{t}{\sqrt{5}} + C = \frac{1}{2\sqrt{5}} \arctan \frac{e^{2x}}{\sqrt{5}} + C.$$

1345. Find the integral $\int \frac{\sin\sqrt{x} + \cos\sqrt{x}}{\sqrt{x}\cdot\sin 2\sqrt{x}}\,dx.$

Solution. Putting $\sqrt{x} = t$, $x = t^2$, $dx = 2t\,dt$, we get

$$\int \frac{\sin\sqrt{x} + \cos\sqrt{x}}{\sqrt{x}\cdot\sin 2\sqrt{x}}\,dx = \int \frac{(\sin t + \cos t)\cdot 2t}{t\sin 2t}\,dt =$$

$$= \int \frac{\sin t + \cos t}{\sin t\cdot\cos t}\,dt = \int\left(\frac{1}{\cos t} + \frac{1}{\sin t}\right)dt$$

$$= \ln\left|\tan\left(\frac{t}{2} + \frac{\pi}{4}\right)\right| + \ln\left|\tan\frac{t}{2}\right| + C$$

(see formulas XXII and XXIII).

Returning to the old variable, we get

$$\int \frac{(\sin\sqrt{x} + \cos\sqrt{x})}{\sqrt{x}\cdot\sin 2\sqrt{x}}\,dx = \ln\left|\tan\left(\frac{\sqrt{x}}{2} + \frac{\pi}{4}\right)\right| + \ln\left|\tan\frac{\sqrt{x}}{2}\right| + C.$$

Find the following integrals:

1346. $\int \frac{e^{\sqrt{2x-1}}}{\sqrt{2x-1}}\,dx.$ **1347.** $\int x^3(1 - 2x^4)^3 dx.$

1348. $\int \sin(2 - 3x)dx.$ **1349.** $\int x\cosh(5x^2 + 3)dx.$

1350. $\int \frac{dx}{(x+1)\sqrt{x}}$. **1351.** $\int x(x^2 + 1)^{3/2}dx.$

1352. $\int \frac{x\,dx}{x^2 - 1}.$ **1353.** $\int \frac{x\,dx}{\sqrt{x^4 - 1}}.$

1354. $\int \frac{\sin 4x\,dx}{\cos^4 2x + 4}.$ **1355.** $\int \frac{dx}{(x-7)\sqrt{x}}$.

1356. $\int \frac{e^{x/2}dx}{\sqrt{16 - e^x}}$. **1357.** $\int \frac{\sqrt{2 - x^2} + \sqrt{2 + x^2}}{\sqrt{4 - x^4}}\,dx.$

Hint. Represent the integral as a sum of integrals.

1358. $\int \frac{x^2dx}{\sqrt{2 - 3x^3}}.$ **1359.** $\int \frac{5x + 3}{\sqrt{3 - x^2}}\,dx.$

1360. $\int \frac{dx}{x^2 - 6x + 25}.$ **1361.** $\int \frac{\sqrt{3x + 5}}{x}\,dx.$

1362. $\int \frac{x\,dx}{2x^4 + 5}.$

9.1.3. Integration by parts. *Integration by parts* is the calculation of an integral by the formula

$$\int u\,dv = uv - \int v\,du,$$

where $u = \varphi(x)$, $u = \psi(x)$ are continuously differentiable functions of x. With the aid of this formula, calculation of the integral $\int u dv$ reduces to finding another integral, $\int v du$; its application is expedient in the cases when the latter integral is either simpler than the former or is identical to it.

Then a function which becomes simpler under differentiation is taken as u and a part of the element of integration whose integral is known or can be found is taken as dv.

Thus, for instance, for integrals of the form $\int P(x)e^{ax}dx$, $\int P(x)\sin axdx$, $\int P(x)\cos axdx$, where $P(x)$ is a polynomial, it is advisable to take $P(x)$ as u and the expressions $e^{ax}dx$, $\sin axdx$, $\cos axdx$, respectively, as dv; for integrals of the form $\int P(x)\ln xdx$, $\int P(x) \arcsin xdx$, $\int P(x)\arccos xdx$, the functions $\ln x$, $\arcsin x$, $\arccos x$, respectively, should be taken as u and the expression $P(x)dx$ as dv.

1363. Find the integral $\int \ln xdx$.

Solution. We put $u = \ln x$, $dv = dx$; then $v = x$, $du = \dfrac{dx}{x}$. Employing the formula for integration by parts, we obtain

$$\int \ln xdx = x\ln x - \int x\frac{dx}{x} = x\ln x - \int dx = x\ln x - x + C = x(\ln x - 1) + C.$$

1364. Find the integral $\int \arctan xdx$.

Solution. Assume $u = \arctan x$, $dv = dx$; then $du = \dfrac{dx}{1 + x^2}$, $v = x$. We get by the formula for integration by parts

$$\int \arctan xdx = x\arctan x - \int \frac{xdx}{1 + x^2} = x\arctan x - \frac{1}{2}\ln(1 + x^2) + C.$$

1365. Find the integral $\int x\sin xdx$.

Solution. We put $u = x$, $dv = \sin xdx$; then $du = dx$, $v = -\cos x$. Hence, we have

$$\int x\sin xdx = -x\cos x + \int \cos xdx = -x\cos x + \sin x + C.$$

Should we choose some other expressions for u and dv, say, $u = \sin x$, $dv = xdx$, we would have got $du = \cos xdx$, $v = (1/2)x^2$, whence

$$\int x\sin xdx = \frac{1}{2}x^2\sin x - \int \frac{1}{2}x^2\cos xdx = \frac{1}{2}x^2\sin x - \frac{1}{2}\int x^2\cos xdx,$$

and we would have arrived at an integral more complex than the original one, since the power of the factor before the trigonometric function has become higher by unity.

1366. Find the integral $\int x^2e^xdx$.

Solution. We put $u = x^2$, $dv = e^xdx$; then $du = 2xdx$, $v = e^x$. We apply the formula for integration by parts:

$$\int x^2e^xdx = x^2e^x - 2\int xe^xdx.$$

We have lowered the degree of x by unity. To find $\int xe^x dx$, we shall once again apply integration by parts. Putting $u = x, dv = e^x dx$, we obtain $du = dx, u = e^x$ and

$$\int x^2e^x dx = x^2e^x - 2(xe^x - \int e^x dx)$$
$$= x^2e^x - 2xe^x + 2e^x + C = e^x(x^2 - 2x + 2) + C.$$

1367. Find the integral $I = \int e^x \sin x\, dx$.

Solution. Assume $u = e^x$, $dv = \sin x dx$; then $du = e^x dx$, $v = -\cos x$. Consequently,

$$I = -e^x \cos x + \int e^x \cos x dx.$$

It seems we have not attained our aim by integrating by parts since we have not simplified the integral. Let us, however, apply integration by parts once again. Assuming $u = e^x$, $dv = \cos x dx$, whence $du = e^x dx$, $v = \sin x$, we get

$$I = -e^x \cos x + (e^x \sin x - I), \text{ i.e. } I = -e^x \cos x + e^x \sin x - I.$$

Applying twice integration by parts, we again obtain the original integral on the right-hand side of the equation. Thus, we arrive at an equation with an unknown integral I, from which we find

$$2I = -e^x \cos x + e^x \sin x, \text{ i.e. } I = \frac{e^x}{2}(\sin x - \cos x) + C.$$

In the final result we have added an arbitrary constant to the primitive we have obtained.

1368. Find the integral $\int \sqrt{a^2 - x^2}dx$, if $a > 0$.

Solution. We put $u = \sqrt{a^2 - x^2}$, $dv = dx$, whence $du = -\dfrac{xdx}{\sqrt{a^2 - x^2}}$, $v = x$. Consequently,

$$\int \sqrt{a^2 - x^2}dx = x\sqrt{a^2 - x^2} - \int \frac{-x^2dx}{\sqrt{a^2 - x^2}} = x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2 - a^2}{\sqrt{a^2 - x^2}}dx,$$

or

$$\int \sqrt{a^2 - x^2}dx = x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}dx + a^2 \arcsin \frac{x}{a}.$$

Hence it follows that

$$2\int \sqrt{a^2 - x^2}dx = x\sqrt{a^2 - x^2} + a^2 \arcsin \frac{x}{a},$$

i.e.

$$\int \sqrt{a^2 - x^2}dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin \frac{x}{a} + C.$$

1369. Derive the recurrence relation for the integral $\displaystyle\int \frac{dx}{(x^2 + a^2)^n}$.

Solution. The integral can be transformed as follows:

$$I_n = \int \frac{dx}{(x^2 + a^2)^n} = \frac{1}{a^2}\int \frac{a^2 + x^2 - x^2}{(x^2 + a^2)^n}dx = \frac{1}{a^2}\int \frac{dx}{(x^2 + a^2)^{n-1}}$$

$$-\frac{1}{a^2}\int\frac{x\cdot xdx}{(x^2+a^2)^n}=\frac{1}{a^2}\cdot I_{n-1}-\frac{1}{a^2}\int x\cdot\frac{xdx}{(x^2+a^2)^n}.$$

Let us put $u = x$, $dv = \dfrac{xdx}{(x^2+a^2)^n}$; then $du = dx$, and

$$v=\frac{1}{2}\int(x^2+a^2)^{-n}\cdot d(x^2+a^2)=-\frac{1}{2(n-1)}\cdot\frac{1}{(x^2+a^2)^{n-1}},$$

whence

$$I_n=\frac{1}{a^2}I_{n-1}+\frac{1}{a^2}\left[\frac{x}{2(n-1)(x^2+a^2)^{n-1}}-\frac{1}{2(n-1)}\cdot\int\frac{dx}{(x^2+a^2)^{n-1}}\right],$$

or

$$I_n=\frac{1}{a^2}\cdot I_{n-1}+\frac{x}{2a^2(n-1)(x^2+a^2)^{n-1}}-\frac{1}{2a^2(n-1)}I_{n-1},$$

that is

$$I_n=\frac{x}{2a^2(n-1)(x^2+a^2)^{n-1}}+\frac{1}{a^2}\frac{2n-3}{2n-2}\cdot I_{n-1}.$$

Putting $n = 2$, we obtain an expression for the integral I_2 in terms of the elementary functions. Assuming now $n = 3$, we find the integral I_3 (the integral I_2 having been found). In this way we can find I_n for any integral positive n.

Find the following integrals:

1370. $\int x\ln x dx$.

1371. $\int \arcsin x dx$.

1372. $\int x^2\arctan x dx$.

1373. $\int (x+1)e^x dx$.

1374. $\int x^2\sin x dx$.

1375. $\int x^5 e^{x^2} dx$.

Hint. Put $x^2 = t$.

1376. $\int (x^2+2x+3)\cos x dx$.

1377. $\int e^{2x}\cos x dx$.

1378. $\int \sin\ln x dx$.

1379. $\int \sin\sqrt{x} dx$.

Hint. Put $\sqrt{x} = t$.

9.2. Integration of Rational Fractions

9.2.1. Integration of partial fractions. A *rational fraction* is a fraction of the form $P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials. A rational fraction is said to be *proper* if the polynomial $P(x)$ is of lower degree than the polynomial $Q(x)$; otherwise the fraction is said to be *improper*.

The term *partial (elementary)* fractions is used for proper fractions of the following types:

I. $\dfrac{A}{x-a}$;

II. $\dfrac{A}{(x-a)^m}$, where m is a positive integer;

III. $\dfrac{Ax+B}{x^2+px+q}$, where $\dfrac{p^2}{4} - q < 0$, that is the quadratic trinomial $x^2 + px + q$ does not possess real roots;

IV. $\dfrac{Ax+B}{(x^2+px+q)^n}$, where n is a positive integer, and the quadratic trinomial $x^2 + px + q$ does not possess real roots.

In all the four cases it is assumed that A, B, p, q, a are real numbers. The fractions we have enumerated will be called, respectively, partial fractions of types I, II, III and IV.

Let us consider the integrals of the partial fractions of the first three types. We have

I. $\displaystyle\int \frac{A}{x-a}\,dx = A\ln|x-a| + C;$

II. $\displaystyle\int \frac{A\,dx}{(x-a)^m} = -\frac{A}{m-1}\cdot\frac{1}{(x-a)^{m-1}} + C;$

III. $\displaystyle\int \frac{dx}{x^2+px+q} = \frac{2}{\sqrt{4q-p^2}}\arctan\frac{2x+p}{\sqrt{4q-p^2}} + C.$

Indeed, for this special case of the partial fraction of type III we obtain

$$x^2 + px + q = \left(x + \frac{p}{2}\right)^2 + q - \frac{p^2}{4}, \quad \text{or} \quad x^2 + px + q = t^2 + a^2,$$

where $t = x + \dfrac{p}{2}$, $a = \dfrac{\sqrt{4q-p^2}}{2}$ (here $\dfrac{p^2}{4} - q < 0$), whence

$$\int \frac{dx}{x^2+px+q} = \int \frac{dt}{t^2+a^2} = \frac{1}{a}\arctan\frac{t}{a} + C$$
$$= \frac{2}{\sqrt{4q-p^2}}\arctan\frac{2x+p}{\sqrt{4q-p^2}} + C.$$

1380. Find the integral $\displaystyle\int \frac{dx}{x^2+6x+25}$.

Solution. We have

$$\int \frac{dx}{x^2+6x+25} = \int \frac{dx}{(x+3)^2+16} = \int \frac{d(x+3)}{(x+3)^2+16} = \frac{1}{4}\arctan\frac{x+3}{4} + C.$$

1381. Find the integral $\displaystyle\int \frac{dx}{2x^2-2x+3}$.

Solution. We have

$$\int \frac{dx}{2x^2 - 2x + 3} = \frac{1}{2}\int \frac{dx}{x^2 - x + \frac{3}{2}} = \frac{1}{2}\int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{3}{2} - \frac{1}{4}\right)} =$$

$$= \frac{1}{2}\int \frac{d\left(x - \frac{1}{2}\right)}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{5}}{2}\right)^2} = \frac{1}{2}\cdot\frac{2}{\sqrt{5}}\cdot\arctan\frac{x - \frac{1}{2}}{\sqrt{5/2}} + C = \frac{1}{\sqrt{5}}\cdot\arctan\frac{2x - 1}{\sqrt{5}} + C.$$

We shall show now how to integrate, in a general form, partial fractions of type III.

It is required to find $\int \frac{Ax + B}{x^2 + px + q}\,dx, \frac{p^2}{4} - q < 0$. Let us isolate the derivative of the denominator from the numerator of the fraction. To do that, we shall represent the numerator in the form

$$Ax + B = (2x + p)\cdot\frac{A}{2} - \frac{Ap}{2} + B.$$

Then

$$\int \frac{Ax + B}{x^2 + px + q}\,dx = \frac{A}{2}\int \frac{2x + p}{x^2 + px + q}\,dx + \left(B - \frac{Ap}{2}\right)\int \frac{dx}{x^2 + px + q}.$$

In the first integral the numerator is the derivative of the denominator; therefore,

$$\int \frac{2x + p}{x^2 + px + q}\,dx = \ln(x^2 + px + q) + C,$$

since $x^2 + px + q > 0$ for any value of x. The second integral, as has been indicated, can be found by the formula

$$\int \frac{dx}{x^2 + px + q} = \frac{2}{\sqrt{4q - p^2}}\arctan\frac{2x + p}{\sqrt{4q - p^2}} + C.$$

Thus, we have

$$\int \frac{Ax + B}{x^2 + px + q}\,dx = \frac{A}{2}\ln(x^2 + px + q) + \frac{2B - Ap}{\sqrt{4q - p^2}}\arctan\frac{2x + p}{\sqrt{4q - p^2}} + C.$$

1382. Find the integral $\int \frac{3x - 1}{x^2 - 4x + 8}\,dx.$

Solution. We have

$$\int \frac{3x - 1}{x^2 - 4x + 8}\,dx = \int \frac{\frac{3}{2}(2x - 4) - 1 + 6}{x^2 - 4x + 8}\,dx =$$

$$= \frac{3}{2}\int \frac{2x-4}{x^2-4x+8}dx + 5\int \frac{dx}{x^2-4x+8} = \frac{3}{2}\ln(x^2-4x+8) + 5\int \frac{dx}{(x-2)^2+2^2}$$

$$= \frac{3}{2}\ln(x^2-4x+8) + \frac{5}{2}\arctan\frac{x-2}{2} + C.$$

1383. Find the integral $\int \frac{xdx}{2x^2+2x+5}$.

Solution. We have

$$\int \frac{xdx}{2x^2+2x+5} = \int \frac{\frac{1}{4}(4x+2) - \frac{1}{2}}{2x^2+2x+5}dx = \frac{1}{4}\int \frac{4x+2}{2x^2+2x+5}dx - \frac{1}{2}\int \frac{dx}{2x^2+2x+5}$$

$$= \frac{1}{4}\ln(2x^2+2x+5) - \frac{1}{2}\cdot\frac{1}{2}\int \frac{dx}{x^2+x+\frac{5}{2}}$$

$$= \frac{1}{4}\ln(2x^2+2x+5) - \frac{1}{4}\int \frac{d\left(x+\frac{1}{2}\right)}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2}$$

$$= \frac{1}{4}\ln(2x^2+2x+5) - \frac{1}{4}\cdot\frac{1}{3/2}\arctan\frac{x+1/2}{3/2} + C$$

$$= \frac{1}{4}\ln(2x^2+2x+5) - \frac{1}{6}\arctan\frac{2x+1}{3} + C.$$

1384. Find the integral $\int \frac{2x^3+3x}{x^4+x^2+1}dx$.

Solution. In this integral we shall first perform the change of variable $x^2 = t$, then we get $2xdx = dt$, $xdx = (1/2)dt$. Consequently,

$$\int \frac{(2x^2+3)xdx}{x^4+x^2+1} = \frac{1}{2}\int \frac{(2t+3)dt}{t^2+t+1} = \frac{1}{2}\int \frac{(2t+1)+2}{t^2+t+1}dt = \frac{1}{2}\int \frac{2t+1}{t^2+t+1}dt$$

$$+ \int \frac{dt}{t^2+t+1} = \frac{1}{2}\ln(t^2+t+1) + \int \frac{d\left(t+\frac{1}{2}\right)}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{2}\ln(t^2+t+1) + \frac{2}{\sqrt{3}}\arctan\frac{t+1/2}{\sqrt{3}/2} + C$$

$$= \frac{1}{2}\ln(x^4+x^2+1) + \frac{2}{\sqrt{3}}\arctan\frac{2x^2+1}{\sqrt{3}} + C.$$

Let us now consider a special case of the integral of a partial fraction of type IV.

For the integral $I_n = \int \frac{dt}{(t^2 + a^2)^n}$ (n being a positive integer), the following recurrence formula holds true:

$$I_n = \frac{1}{2a^2(n-1)} \cdot \frac{t}{(t^2 + a^2)^{n-1}} + \frac{1}{a^2} \cdot \frac{2n-3}{2n-2} \cdot I_{n-1}.$$

After $(n - 1)$ applications, this formula makes it possible to reduce the given integral I_n to the tabular integral $\int \frac{dt}{t^2 + a^2}$.

1385. Find the integral $I_3 = \int \frac{dx}{(x^2 + 1)^3}$.

Solution. Here $n = 3$. After the first application of the recurrence formula, we get

$$I_3 = \frac{1}{2\cdot(3-1)} \cdot \frac{x}{(x^2+1)^{3-1}} + \frac{2\cdot 3 - 3}{2\cdot 3 - 2} \cdot I_{3-1} = \frac{1}{4}\frac{x}{(x^2+1)^2} + \frac{3}{4} I_2.$$

To the integral $I_2 = \int \frac{dx}{(x^2 + 1)^2}$ we again apply the recurrence formula (setting $n = 2$ here):

$$I_2 = \frac{1}{2(2-1)} \cdot \frac{x}{(x^2+1)^{2-1}} + \frac{2\cdot 2 - 3}{2\cdot 2 - 2} I_{2-1} = \frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2} I_1$$

$$= \frac{x}{2(x^2+1)} + \frac{1}{2}\int \frac{dx}{x^2+1} = \frac{x}{2(x^2+1)} + \frac{1}{2} \arctan x + C.$$

Thus, we get

$$\int \frac{dx}{(x^2+1)^3} = \frac{x}{4(x^2+1)^2} + \frac{3}{4}\left[\frac{x}{2(x^2+1)} + \frac{1}{2} \arctan x\right] + C.$$

The final result is

$$\int \frac{dx}{(x^2+1)^3} = \frac{x}{4(x^2+1)^2} + \frac{3x}{8(x^2+1)} + \frac{3}{8} \arctan x + C.$$

We shall show now how to integrate partial fractions of type IV in a general case. It is required to find the integral $\int \frac{Ax + B}{(x^2 + px + q)^n} dx$, $\frac{p^2}{4} - q < 0$.

Let us isolate in the numerator the derivative of the quadratic trinomial appearing in the denominator:

$$\int \frac{Ax + B}{(x^2 + px + q)^n}\,dx = \int \frac{\frac{A}{2}(2x + p) + \left(B - \frac{Ap}{2}\right)}{(x^2 + px + q)^n}\,dx$$

$$= \frac{A}{2}\int \frac{2x + p}{(x^2 + px + q)^n}\,dx + \left(B - \frac{Ap}{2}\right)\cdot \int \frac{dx}{(x^2 + px + q)^n}.$$

While the first integral on the right-hand side of the equation can be easily found by means of the substitution $x^2 + px + q = t$, the second integral must be transformed as follows:

$$\int \frac{dx}{(x^2 + px + q)^n} = \int \frac{dx}{\left[\left(x + \frac{p}{2}\right)^2 + \left(q - \frac{p^2}{4}\right)\right]^n}.$$

Setting now $x + \frac{p}{2} = t$, $dx = dt$ and introducing the designation $q - \frac{p^2}{4} = a^2$, we get $\int \frac{dx}{(x^2 + px + q)^n} = \int \frac{dt}{(t^2 + a^2)^n}$.

Thus we see that integration of an elementary fraction of type IV can be carried out with the aid of the recurrence formula.

1386. Find the integral $\int \frac{3x + 2}{(x^2 + 2x + 10)^2}\,dx$.

Solution. We have

$$\int \frac{3x + 2}{(x^2 + 2x + 10)^2}\,dx = \int \frac{\frac{3}{2}(2x + 2) + (2 - 3)}{(x^2 + 2x + 10)^2}\,dx$$

$$= \frac{3}{2}\int \frac{2x + 2}{(x^2 + 2x + 10)^2}\,dx - \int \frac{dx}{[(x + 1)^2 + 9]^2}.$$

We perform the change of variable $x^2 + 2x + 10 = z$, $(2x + 2)dx = dz$ in the first integral and put $x + 1 = t$, $dx = dt$ in the second integral. We obtain

$$\int \frac{3x + 2}{(x^2 + 2x + 10)^2}\,dx = \frac{3}{2}\int \frac{dz}{z^2} - \int \frac{dt}{(t^2 + 9)^2} = \frac{3}{2}\int z^{-2}dz - \int \frac{dt}{(t^2 + 9)^2}$$

$$= -\frac{3}{2}z^{-1} - \left[\frac{1}{2(2 - 1)\cdot 9}\cdot \frac{t}{(t^2 + 9)^{2-1}} + \frac{1}{9}\cdot \frac{2\cdot 2 - 3}{2\cdot 2 - 2}\cdot \int \frac{dt}{t^2 + 9}\right]$$

$$= -\frac{3}{2z} - \frac{1}{18}\cdot \frac{t}{t^2 + 9} - \frac{1}{18}\cdot \frac{1}{3}\arctan\frac{t}{3} + C.$$

Returning now to the old variable, we get

$$\int \frac{3x+2}{(x^2+2x+10)^2}dx = -\frac{3}{2(x^2+2x+10)} - \frac{1}{18}\cdot\frac{x+1}{(x+1)^2+9} - \frac{1}{54}\arctan\frac{x+1}{3} + C$$

$$= -\frac{3}{2(x^2+2x+10)} - \frac{1}{18}\cdot\frac{x+1}{x^2+2x+10} - \frac{1}{54}\arctan\frac{x+1}{3} + C.$$

Find the following integrals:

1387. $\int \frac{dx}{(x-1)^4}$. **1388.** $\int \frac{dx}{(2x+3)^3}$.

1389. $\int \frac{dx}{x^2-6x+18}$. **1390.** $\int \frac{x^2dx}{x^6+2x^3+3}$.

1391. $\int \frac{x-2}{x^2-4x+7}dx$. 1392. $\int \frac{5x+3}{x^2+10x+29}dx$.

1393. $\int \frac{x+1}{5x^2+2x+1}dx$. 1394. $\int \frac{dx}{(x^2+2)^3}$.

1395. $\int \frac{2x+3}{(x^2+2x+5)^2}dx$.

9.2.2. Integrating a rational fraction by means of decomposition into partial fractions. Before integrating the rational fraction $P(x)/Q(x)$, we must perform the following algebraic transformations and calculations:

(1) if we are given an improper rational fraction, we must isolate its integral part, that is, represent it in the form

$$\frac{P(x)}{Q(x)} = M(x) + \frac{P_1(x)}{Q(x)},$$

where $M(x)$ is a polynomial and $P_1(x)/Q(x)$ is a proper rational fraction;

(2) factor the denominator of the fraction into linear and quadratic multipliers:

$$Q(x) = (x-a)^m \dots (x^2+px+q)^n \dots,$$

where $\frac{p^2}{4} - q < 0$, that is, the trinomial x^2+px+q possesses complex conjugate roots;

(3) decompose the proper rational fraction into partial fractions:

$$\frac{P_1(x)}{Q(x)} = \frac{A_1}{(x-a)^m} + \frac{A_2}{(x-a)^{m-1}} + \dots + \frac{A_m}{x-a} + \dots$$

$$\dots + \frac{B_1x+C_1}{(x^2+px+q)^n} + \frac{B_2x+C_2}{(x^2+px+q)^{n-1}} + \dots + \frac{B_nx+C_n}{x^2+px+q} + \dots;$$

(4) compute the undetermined coefficients $A_1, A_2, \ldots, A_m, \ldots, B_1, C_1, B_2, C_2, \ldots, B_n, C_n, \ldots$, for which purpose it is necessary to reduce the latter equality to the common denominator, equate the coefficients in the same degrees of x on the right-hand and left-hand sides of the identity obtained and solve the system of linear equations with respect to the sought-for coefficients. Another method can also be used to determine the coefficients, which consists in assigning arbitrary numerical values to the variable x in the identity obtained. It is sometimes useful to combine the two methods of calculation of the coefficients. As a result, integration of a rational fraction will reduce to finding integrals of a polynomial and of partial rational fractions.

Case 1. The denominator possesses only distinct real roots, that is, can be factored into nonrepeating multipliers of the first degree.

1396. Find the integral $\int \frac{x^2 + 2x + 6}{(x-1)(x-2)(x-4)}\, dx.$

Solution. Since each of the binomials $x - 1$, $x - 2$, $x - 4$ appearing in the denominator is of the first degree, the given proper rational fraction can be represented as a sum of partial fractions of type I:

$$\frac{x^2 + 2x + 6}{(x-1)(x-2)(x-4)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-4}.$$

Clearing the equation of fractions, we get

$$x^2 + 2x + 6 = A(x-2)(x-4) + B(x-1)(x-4) + C(x-1)(x-2). \qquad (*)$$

Consequently,

$$x^2 + 2x + 6 = A(x^2 - 6x + 8) + B(x^2 - 5x + 4) + C(x^2 - 3x + 2).$$

Combining the terms of like degrees, we get

$$x^2 + 2x + 6 = (A + B + C)x^2 + (-6A - 5B - 3C)x + (8A + 4B + 2C).$$

Comparing the coefficients in like powers of x, we get the system of equations

$$\begin{cases} A + B + C = 1, \\ -6A - 5B - 3C = 2, \\ 8A + 4B + 2C = 6, \end{cases}$$

from which we find $A = 3$, $B = -7$, $C = 5$.

Thus, decomposition of a rational fraction into partial fractions has the form

$$\frac{x^2 + 2x + 6}{(x-1)(x-2)(x-4)} = \frac{3}{x-1} - \frac{7}{x-2} + \frac{5}{x-4}.$$

The unknowns A, B, and C in the decomposition could have been determined in another way. After clearing of fractions, x could have been assigned as many particular values as there are unknowns in the system, in the given case three particular values.

It is especially convenient to assign to x the values which are real roots of the denominator. Let us try to solve the given problem by this technique. After clearing of fractions, we have obtained equality (*). The real roots of the denominator are the numbers 1, 2 and 4. Let us set $x = 1$ in this equality. Then

$$1^2 + 2\cdot 1 + 6 = A(1 - 2)(1 - 4) + B(1 - 1)(1 - 4) + C(1 - 1)(1 - 2),$$

whence $9 = 3A$, i.e. $A = 3$. Putting $x = 2$, we get $14 = -2B$, i.e. $B = -7$; putting $x = 4$, we have $30 = 6C$, i.e. $C = 5$. As a result we have obtained the same values as by the first method of determining the unknowns.

Thus we have

$$\int \frac{x^2 + 2x + 6}{(x - 1)(x - 2)(x - 4)}\, dx = 3\int \frac{dx}{1 - x} - 7\int \frac{dx}{x - 2} + 5\int \frac{dx}{x - 4}$$

$$= 3 \ln|x - 1| - 7 \ln|x - 2| + 5 \ln|x - 4| + C = \ln\left|\frac{(x - 1)^3(x - 4)^5}{(x - 2)^7}\right| + C.$$

Case 2. The denominator possesses only real roots, some of them being multiple roots, that is, the denominator can be factored into first-degree multipliers, some of which being repeated.

1397. Find the integral $\displaystyle\int \frac{x^2 + 1}{(x - 1)^3(x + 3)}\, dx.$

Solution. The factor $(x - 1)^3$ is associated with the sum of three partial fractions $\dfrac{A}{(x - 1)^3} + \dfrac{B}{(x - 1)^2} + \dfrac{C}{x - 1}$, and the factor $x + 3$, with the partial fraction $\dfrac{D}{x + 3}$. Thus we have

$$\frac{x^2 + 1}{(x - 1)^3(x + 3)} = \frac{A}{(x - 1)^3} + \frac{B}{(x - 1)^2} + \frac{C}{x - 1} + \frac{D}{x + 3}.$$

Let us clear the equation of fraction

$$x^2 + 1 = A(x + 3) + B(x - 1)(x + 3) + C(x - 1)^2(x + 3) + D(x - 1)^3.$$

The real roots of the denominator are the numbers 1 and -3. Setting $x = 1$, we get $2 = 4A$, i.e. $A = 1/2$. For $x = -3$, we have $10 = -64D$, i.e. $D = -5/32$.

Let us compare now the coefficients in the leading power of x, i.e. in x^3. On the left-hand side, there is no term with x^3, that is, the coefficient in x^3 is zero. On the right-hand side, the coefficient in x^3 is equal to $C + D$. Thus, $C + D = 0$, whence $C = 5/32$.

It remains to determine the coefficient B. For that purpose it is necessary to have one more equation. It can be obtained by comparing the coefficients in like powers of x (say, x^2) or by assigning to x some numerical value. It is more convenient to take a value for which the computations will be easier. Putting $x = 0$, for instance,

we get

$$1 = 3A - 3B + 3C - D, \quad \text{or} \quad 1 = \frac{3}{2} - 3B + \frac{15}{32} + \frac{5}{32}, \text{ i.e. } B = \frac{3}{8}.$$

The final decomposition of the given fraction into partial fractions has the form

$$\frac{x^2 + 1}{(x - 1)^3(x + 3)} = \frac{1}{2(x - 1)^3} + \frac{3}{8(x - 1)^2} + \frac{5}{32(x - 1)} + \frac{5}{32(x + 3)}.$$

Thus we obtain

$$\int \frac{x^2 + 1}{(x - 1)^3(x + 3)}\,dx = \frac{1}{2}\int \frac{dx}{(x - 1)^3} + \frac{3}{8}\int \frac{dx}{(x - 1)^2} + \frac{5}{32}\int \frac{dx}{x - 1} - \frac{5}{32}\int \frac{dx}{x + 3}$$

$$= -\frac{1}{4(x - 1)^2} - \frac{3}{8(x - 1)} + \frac{5}{32}\ln\left|\frac{x - 1}{x + 3}\right| = C.$$

Case 3. There are one-fold complex roots among the roots of the denominator, that is, the factorization of the denominator contains quadratic nonrepeating factors.

1398. Find the integral $\displaystyle\int \frac{dx}{x^5 - x^2}$.

Solution. Let us factor the denominator:

$$x^5 - x^2 = x^2(x^3 - 1) = x^2(x - 1)(x^2 + x + 1).$$

Then

$$\frac{1}{x^5 - x^2} = \frac{1}{x^2(x - 1)(x^2 + x + 1)} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x - 1} + \frac{Dx + E}{x^2 + x + 1}.$$

We clear the equations of fractions:

$$1 = A(x - 1)(x^2 + x + 1) + Bx(x - 1)(x^2 + x + 1) + \\ + Cx^2(x^2 + x + 1) + (Dx + E)x^2(x - 1).$$

The real roots of the denominator are the numbers 0 and 1.

For $x = 0$ we have $1 = -A$, i.e. $A = -1$.

For $x = 1$ we have $1 = 3C$, i.e. $C = 1/3$.

We rewrite the previous equality in the form

$$1 = A(x^3 - 1) + B(x^4 - x) + C(x^4 + x^3 + x^2) + Dx^4 + Ex^3 - Dx^3 - Ex^2.$$

Comparing the coefficients in x^4, x^3, x^2, we arrive at the system of equations

$$\begin{cases} B + C + D = 0, \\ A + C + E - D = 0, \\ C - E = 0, \end{cases}$$

from which we find: $B = 0$, $D = -1/3$, $E = 1/3$. Thus we have

$$\frac{1}{x^5 - x^2} = -\frac{1}{x^2} + \frac{1}{3(x-1)} - \frac{x-1}{3(x^2+x+1)}.$$

Consequently,

$$\int \frac{dx}{x^5 - x^2} = -\int \frac{dx}{x^2} + \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{x-1}{x^2+x+1}\,dx$$

$$= \frac{1}{x} + \frac{1}{3}\ln|x-1| - \frac{1}{6}\int \frac{2x+1-3}{x^2+x+1}\,dx$$

$$= \frac{1}{x} + \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) + \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$= \frac{1}{x} + \frac{1}{6}\ln\frac{(x-1)^2}{x^2+x+1} + \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + C.$$

Case 4. There are multiple complex roots among the roots of the denominator, that is, the factorization of the denominator contains repeating quadratic factors.

1399. Find the integral $\int \frac{x^3 - 2x}{(x^2+1)^2}\,dx$.

Solution. Since $x^2 + 1$ is a factor of multiplicity 2, it follows that

$$\frac{x^3 - 2x}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)^2} + \frac{Cx+D}{x^2+1}.$$

Clearing the equation, we obtain

$$x^3 - 2x = Ax + B + (Cx+D)(x^2+1).$$

We equate the coefficients in like powers of x:

$$\begin{array}{c|l} x^3 & 1 = C, \\ x^2 & 0 = D, \\ x & -2 = A + C;\ A = -3, \\ x^0 & 0 = B + D;\ B = 0. \end{array}$$

Consequently,

$$\int \frac{x^3 - 2x}{(x^2+1)^2}\,dx = \int \frac{-3x\,dx}{(x^2+1)^2} + \int \frac{x\,dx}{x^2+1} = -\frac{3}{2}\int \frac{d(x^2+1)}{(x^2+1)^2} + \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1}$$

$$= \frac{3}{2(x^2+1)} + \frac{1}{2}\ln(x^2+1) + C.$$

The given integral could have been found in a simpler way, by means of the substitution $x^2 + 1 = t$.

1400. Find the integral $\displaystyle\int \frac{x^3 + 3x^2 + 5x + 7}{x^2 + 2}\,dx$.

Solution. Let us isolate the integral part of the given improper rational fraction:

$$\begin{array}{r|l} x^3 + 3x^2 + 5x + 7 & x^2 + 2 \\ \cline{2-2} -\;\; x^3 \qquad\quad + 2x \qquad & x + 3 + \dfrac{3x + 1}{x^2 + 2} \\ \cline{1-1} 3x^2 + 3x + 7 & \\ -\;\; 3x^2 \qquad\quad + 6 & \\ \cline{1-1} 3x + 1. & \end{array}$$

It follows that

$$\frac{x^3 + 3x^2 + 5x + 7}{x^2 + 2} = x + 3 + \frac{3x + 1}{x^2 + 2}.$$

Hence we find

$$\int \frac{x^3 + 3x^2 + 5x + 7}{x^2 + 2}\,dx = \int \left(x + 3 + \frac{3x + 1}{x^2 + 2}\right)dx$$

$$= \int x\,dx + 3\int dx + \int \frac{3x + 1}{x^2 + 2}\,dx = \frac{1}{2}x^2 + 3x + \frac{3}{2}\int \frac{2x\,dx}{x^2 + 2} + \int \frac{dx}{x^2 + 2}$$

$$= \frac{1}{2}x^2 + 3x + \frac{3}{2}\ln(x^2 + 2) + \frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}} + C.$$

1401. Find the integral $\displaystyle\int \frac{x + 4}{x^3 + 6x^2 + 11x + 6}\,dx$.

Solution. Since the integrand is a proper fraction, it should be immediately represented as a sum of partial fractions. It is easy to see that the polynomial $x^3 + 6x^2 + 11x + 6$ vanishes at $x = -1$ and, therefore, it is exactly divisible by $x + 1$.

Let us carry out the division:

$$\begin{array}{r|l} x^3 + 6x^2 + 11x + 6 & x + 1 \\ \cline{2-2} -\;\; x^3 + \;\; x^2 \qquad\qquad\quad & x^2 + 5x + 6 \\ \cline{1-1} 5x^2 + 11x \qquad & \\ -\;\; 5x^2 + \;\; 5x \qquad & \\ \cline{1-1} 6x + 6 & \\ -\;\; 6x + 6 & \\ \cline{1-1} \end{array}$$

Consequently,

$$x^3 + 6x^2 + 11x + 6 = (x + 1)(x^2 + 5x + 6) = (x + 1)(x + 2)(x + 3),$$

$$\frac{x+4}{x^3+6x^2+11x+6}=\frac{x+4}{(x+1)(x+2)(x+3)}=\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{x+3}.$$

Clearing of fractions, we get

$$x+4=A(x+2)(x+3)+B(x+1)(x+3)+C(x+1)(x+2).$$

Setting $x=-1$, we find $3=2A$, i.e. $A=3/2$. If $x=-2$, we get $2=-B$, i.e. $B=-2$. For $x=-3$, we get $1=2C$, i.e. $C=1/2$.

Thus we have

$$\int\frac{x+4}{x^3+6x^2+11x+6}\,dx=\frac{3}{2}\int\frac{dx}{x+1}-2\int\frac{dx}{x+2}+\frac{1}{2}\int\frac{dx}{x+3}$$

$$=\frac{3}{2}\ln|x+1|-2\ln|x+2|+\frac{1}{2}\ln|x+3|+C.$$

1402. Find the integral $\displaystyle\int\frac{x^5+1}{x^4+8x^2+16}\,dx.$

Solution. First of all, we should isolate the integral part:

$$\begin{array}{r|l} -\;x^5+1 & x^4-8x^2+16 \\ \cline{2-2} \underline{x^5-8x^3+16x} & x+\dfrac{8x^3-16x+1}{x^4-8x^2+16} \\ 8x^3-16x+1 & \end{array}$$

We obtain

$$\frac{x^5+1}{x^4-8x^2+16}=x+\frac{8x^3-16x+1}{x^4-8x^2+16}=x+\frac{8x^3-16x+1}{(x-2)^2(x+2)^2}.$$

Now we decompose the proper fraction into partial fractions:

$$\frac{8x^3-16x+1}{(x-2)^2(x+2)^2}=\frac{A}{(x-2)^2}+\frac{B}{x-2}+\frac{C}{(x+2)^2}+\frac{D}{x+2},$$

and clear of fractions:

$$8x^3-16x+1=A(x+2)^2+B(x-2)(x+2)^2+C(x-2)^2+D(x-2)^2(x+2).$$

Setting $x=2$, we find $33=16A$, i.e. $A=33/16$.
For $x=-2$, we get $-31=16C$, i.e. $C=-31/16$.
If $x=0$, then $1=4A-8B+4C+8D$.
Replacing A and C by their values, we obtain

$$1=\frac{33}{4}-8B-\frac{31}{4}+8D,\quad\text{or}\quad -16B+16D=1.$$

To find B and D, we set up one more equation. Comparing the coefficients in x^3, we get $8=B+D$. Solving the system of equations

$$\begin{cases} B+D=8, \\ -16B+16D=1, \end{cases}$$

we find $D = 129/32$, $B = 127/32$.

Thus it follows that

$$\int \frac{x^5+1}{x^4-8x^2+16}\,dx = \int \left[x + \frac{33/16}{(x-2)^2} + \frac{127/32}{x-2} - \frac{31/16}{(x+2)^2} + \frac{129/32}{x+2}\right]dx$$

$$= \frac{x^2}{2} - \frac{33}{16(x-2)} + \frac{127}{32}\ln|x-2| + \frac{31}{16(x+2)} + \frac{129}{32}\ln|x+2| + C.$$

1403. Find the integral $\int \frac{x^2\,dx}{(x-1)^5}$.

Solution. The integrand is a proper rational fraction, and we could find the integral by representing the fraction as a sum of partial fractions. But we can considerably simplify the calculation of the integral if we make the change of variable $x - 1 = t$; then $x = t + 1$ and $dx = dt$. As a result we obtain

$$\int \frac{x^2\,dx}{(x-1)^5} = \int \frac{(t+1)^2\cdot dt}{t^5} = \int \frac{t^2+2t+1}{t^5}\,dt = \int \frac{dt}{t^3} + 2\int \frac{dt}{t^4} + \int \frac{dt}{t^5}$$

$$= -\frac{1}{2t^2} - \frac{2}{3t^3} - \frac{1}{4t^4} + C = -\frac{1}{2(x-1)^2} - \frac{2}{3(x-1)^3}$$

$$- \frac{1}{4(x-1)^4} + C = -\frac{6x^2-4x+1}{12(x-1)^4} + C.$$

1404. Find the integral $\int \frac{x\,dx}{x^4+6x^2+5}$.

Solution. Let us transform the denominator: $x^4 + 6x + 5 = (x^2+3)^2 - 4$. Now we have

$$\int \frac{x\,dx}{x^4+6x^2+5} = \int \frac{x\,dx}{(x^2+3)^2-4}.$$

We perform the change of variable $x^2 + 3 = t$; then $2x\,dx = dt$ and

$$\int \frac{x\,dx}{x^4+6x^2+5} = \int \frac{x\,dx}{(x^2+3)^2-4} = \frac{1}{2}\int \frac{dt}{t^2-4}$$

$$= \frac{1}{2}\cdot\frac{1}{4}\ln\left|\frac{t-2}{t+2}\right| + C = \frac{1}{8}\ln\frac{x^2+1}{x^2+5} + C.$$

We see from the last two examples that it is sometimes useful to make a change of variable before integrating a rational fraction.

Find the following integrals:

1405. $\int \frac{x+2}{x(x-3)}\,dx.$ **1406.** $\int \frac{2x^2+x+3}{(x+2)(x^2+x+1)}\,dx.$

1407. $\int \frac{5x^3 - 17x^2 + 18x - 5}{(x-1)^3(x-2)}\,dx.$ **1408.** $\int \frac{dx}{x^3 - 8}.$

1409. $\int \frac{x^3 + x + 1}{x^4 - 81}\,dx.$ **1410.** $\int \frac{dx}{(x^2 - 2x)^2}.$

1411. $\int \frac{x + 1}{(x^2 + 1)(x^2 + 9)}\,dx.$ **1412.** $\int \frac{(19/16)x^2 + x + 1}{(x^2 + 4)(x^2 + 2x + 5)}\,dx.$

1413. $\int \frac{x^2 + 2}{x^4 + 4}\,dx.$

Hint. Represent the denominator in the form $x^4 + 4 = (x^2 + 2)^2 - 4x^2$.

1414. $\int \frac{x^2\,dx}{x^2 - 4x + 3}.$ **1415.** $\int \frac{x^3 + x^2}{x^2 - 6x + 5}\,dx.$

1416. $\int \frac{x^4}{x^4 - 16}\,dx.$ **1417.** $\int \frac{3x^3 + x^2 + 5x + 1}{x^3 + x}\,dx.$

9.3. Integration of Basic Irrational Functions

9.3.1. Integrals of the form $\int R(x, (ax + b)^{m_1/n_1}, (ax + b)^{m_2/n_2}, \ldots)dx$, where R is a rational function; $m_1, n_1, m_2, n_2, \ldots$ are integers. By means of the substitution $ax + b = t^s$, where s is the least common multiple of the numbers $n_1, n_2, \ldots$, the indicated integral can be transformed into an integral of a rational function.

1418. Find the integral $I = \int \frac{dx}{(2x + 1)^{2/3} - (2x + 1)^{1/2}}.$

Solution. Here $n_1 = 3$, $n_2 = 2$; therefore, $s = 6$. Let us apply the substitution $2x + 1 = t^6$, then $x = (t^6 - 1)/2$, $dx = 3t^5\,dt$, and it follows that

$$I = \int \frac{3t^5\,dt}{t^4 - t^3} = 3\int \frac{t^2\,dt}{t - 1} = 3\int \frac{t^2 - 1 + 1}{t - 1}\,dt$$

$$= 3\int \left(t + 1 + \frac{1}{t - 1}\right)dt = \frac{3}{2}t^2 + 3t + 3\ln|t - 1| + C.$$

Now we return to the old variable. Since $t = (2x + 1)^{1/6}$, it follows that

$$I = \frac{3}{2}(2x + 1)^{1/3} + 3(2x + 1)^{1/6} + 3\ln|\sqrt[6]{2x + 1} - 1| + C.$$

9.3.2. Integrals of the form $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$. By isolating a complete square from a quadratic trinomial, such integrals reduce to the tabular integrals of the form XX or XXI.

1419. Find the integral $\int \frac{dx}{\sqrt{x^2 + 2x + 5}}$.

Solution. Let us reduce the quadratic trinomial to the form $x^2 + 2x + 5 = (x + 1)^2 + 4$. Then we shall have

$$\int \frac{dx}{\sqrt{x^2 + 2x + 5}} = \int \frac{d(x + 1)}{\sqrt{(x + 1)^2 + 4}} = \ln|x + 1 \sqrt{x^2 + 2x + 5}| + C.$$

1420. Find the integral $\int \frac{dx}{\sqrt{-3x^2 + 4x - 1}}$.

Solution. We have

$$\int \frac{dx}{\sqrt{-3x^2 + 4x - 1}} = \frac{1}{\sqrt{3}} \int \frac{d\left(x - \frac{2}{3}\right)}{\sqrt{\frac{1}{9} - \left(x - \frac{2}{3}\right)^2}}$$

$$= \frac{1}{\sqrt{3}} \arcsin \frac{x - 2/3}{1/3} + C = \frac{1}{\sqrt{3}} \arcsin(3x - 2) + C.$$

9.3.3. Integrals of the form $\int \frac{Ax + B}{\sqrt{ax^2 + bx + c}} dx$. To find this integral, let us isolate in the numerator the derivative of the quadratic trinomial appearing under the root sign and expand the integral into a sum of two integrals:

$$\int \frac{Ax + B}{\sqrt{ax^2 + bx + c}} dx = \int \frac{\frac{A}{2a}(2ax + b) + B - \frac{Ab}{2a}}{\sqrt{ax^2 + bx + c}} dx$$

$$= \frac{A}{2a} \int \frac{d(ax^2 + bx + c)}{\sqrt{ax^2 + bx + c}} + \left(B - \frac{Ab}{2a}\right) \int \frac{dx}{\sqrt{ax^2 + bx + c}}.$$

The first of the integrals we have obtained can be identified with the tabular integral XVII and the second integral has been discussed in 9.3.2.

1421. Find the integral $\int \frac{5x - 3}{\sqrt{2x^2 + 8x + 1}} dx$.

Solution. Let us isolate in the numerator the derivative of the element of integration:

$$\int \frac{5x - 3}{\sqrt{2x^2 + 8x + 1}} dx = \int \frac{\frac{5}{4}(4x + 8) - 13}{\sqrt{2x^2 + 8x + 1}} dx$$

$$= \frac{5}{4} \int \frac{4x + 8}{\sqrt{2x^2 + 8x + 1}} dx - 13 \int \frac{dx}{\sqrt{2x^2 + 8x + 1}}$$

$$= \frac{5}{2}\sqrt{2x^2 + 8x + 1} - \frac{13}{\sqrt{2}}\int \frac{dx}{\sqrt{x^2 + 4x + \frac{1}{2}}}$$

$$= \frac{5}{2}\sqrt{2x^2 + 8x + 1} - \frac{13}{\sqrt{2}}\ln\left|x + 2 + \sqrt{x^2 + 4x + \frac{1}{2}}\right| + C.$$

1422. Find the integral $\int \frac{3x + 4}{\sqrt{-x^2 + 6x - 8}}\,dx.$

Solution. We have

$$\int \frac{3x + 4}{\sqrt{-x^2 + 6x - 8}}\,dx = \int \frac{-\frac{3}{2}(-2x + 6) + 13}{\sqrt{-x^2 + 6x - 8}}\,dx$$

$$= -\frac{3}{2}\int \frac{-2x + 6}{\sqrt{-x^2 + 6x - 8}}\,dx + 13\int \frac{dx}{\sqrt{1 - (x - 3)^2}}$$

$$= -3\sqrt{-x^2 + 6x - 8} + 13\arcsin(x - 3) + C.$$

9.3.4. Integrals of the form $\int \frac{dx}{(x - \alpha)\sqrt{ax^2 + bx + c}}$. With the aid of the substitution $x - \alpha = 1/t$ this integral can be reduced to that discussed in 9.2.

1423. Find the integral $\int \frac{dx}{x\sqrt{5x^2 - 2x + 1}}$.

Solution. We put $x = 1/t$, then $dx = -(1/t^2)dt$ and

$$\int \frac{dx}{x\sqrt{5x^2 - 2x + 1}} = -\int \frac{(dt)/t^2}{(1/t)\sqrt{5/t^2 - 2/t + 1}}$$

$$= -\int \frac{dt}{\sqrt{t^2 - 2t + 5}} = -\ln|t - 1 + \sqrt{t^2 - 2t + 5}| + C$$

$$= -\ln\left|\frac{1}{x} - 1 + \sqrt{\frac{1}{x^2} - \frac{2}{x} + 5}\right| + C = -\ln\left|\frac{1 - x + \sqrt{5x^2 - 2x + 1}}{x}\right| + C.$$

1424. Find the integral $\int \frac{dx}{(x - 1)\sqrt{-x^2 + 2x + 3}}$.

Solution. Putting $x - 1 = 1/t$, we get $x = 1/t + 1$ and $dx = -(1/t^2)\,dt$. Consequently,

$$\int \frac{dx}{(x - 1)\sqrt{-x^2 + 2x + 3}} = -\int \frac{(dt)/t^2}{\frac{1}{t}\sqrt{-\left(1 + \frac{1}{t}\right)^2 + 2\left(1 + \frac{1}{t}\right) + 3}}$$

$$= -\int \frac{dt}{t\sqrt{-1 - \frac{2}{t} - \frac{1}{t^2} + 2 + \frac{2}{t} + 3}} = -\int \frac{dt}{\sqrt{4t^2 - 1}}$$

$$= -\frac{1}{2}\int \frac{dt}{\sqrt{t^2 - \frac{1}{4}}} = -\frac{1}{2}\ln\left|t + \sqrt{t^2 - \frac{1}{2}}\right| + C$$

$$= -\frac{1}{2}\ln\left|\frac{1}{x-1} + \sqrt{\left(\frac{1}{x-1}\right)^2 - \frac{1}{4}}\right| + C = -\frac{1}{2}\ln\left|\frac{2 + \sqrt{-x^2 + 2x + 3}}{2(x-1)}\right| + C.$$

1425. Find the integral $I = \int \frac{3x + 2}{(x+1)\sqrt{x^2 + 3x + 3}}\, dx.$

Solution. Having written the numerator of the integrand in the form $3x + 2 = 3(x + 1) - 1$, we get

$$I = \int \frac{3(x+1) - 1}{(x+1)\sqrt{x^2 + 3x + 3}}\, dx.$$

Let us represent the given integral as a difference between two integrals:

$$I = 3\int \frac{dx}{\sqrt{x^2 + 3x + 3}} - \int \frac{dx}{(x+1)\sqrt{x^2 + 3x + 3}}.$$

Then we apply formula XXI to the first integral and the substitution $x + 1 = 1/t$ to the second integral:

$$I = 3\ln\left|x + \frac{3}{2}\sqrt{x^2 + 3x + 3}\right| + \int \frac{(dt)/t^2}{\frac{1}{t}\sqrt{\left(\frac{1}{t} - 1\right)^2 + 3\left(\frac{1}{t} - 1\right) + 3}}$$

$$= 3\ln\left|x + \frac{3}{2} + \sqrt{x^2 + 3x + 3}\right| + \int \frac{dt}{\sqrt{t^2 + t + 1}}$$

$$= 3\ln\left|x + \frac{3}{2} + \sqrt{x^2 + 3x + 3}\right| + \ln\left|t + \frac{1}{2} + \sqrt{t^2 + t + 1}\right| + C$$

$$= 3\ln\left|x + \frac{3}{2} + \sqrt{x^2 + 3x + 3}\right| + \ln\left|\frac{1}{x+1} + \frac{1}{2} \frac{\sqrt{x^2 + 3x + 3}}{x+1}\right| + C.$$

9.3.5. Integrals of the form $\int \frac{P_n(x)}{\sqrt{ax^2 + bx + c}}\, dx$**, where** $P_n(x)$ **is an** n**th-degree polynomial.** An integral of this form can be found with the aid of the identity

$$\int \frac{P_n(x)}{\sqrt{ax^2 + bx + c}}\,dx$$
$$= Q_{n-1}(x)\sqrt{ax^2 + bx + c} + \lambda \int \frac{dx}{\sqrt{ax^2 + bx + c}},$$

where $Q_{n-1}(x)$ is a polynomial of degree $(n - 1)$ with undetermined coefficients, and λ is a number.

Differentiating the indicated identity and reducing the result to a common denominator, we obtain an equality of two polynomials which yields the values of the coefficients of the polynomial $Q_{n-1}(x)$ and of the number λ.

1426. Find the integral $\displaystyle\int \frac{x^3 + 2x^2 + 3x + 4}{\sqrt{x^2 + 2x + 2}}\,dx.$

Solution. Here $n = 3$, and therefore the corresponding identity has the form

$$\int \frac{x^3 + 2x^2 + 3x + 4}{\sqrt{x^2 + 2x + 2}}\,dx$$
$$= (b_0x^2 + b_1x + b_2)\sqrt{x^2 + 2x + 2} + \lambda \int \frac{dx}{\sqrt{x^2 + 2x + 2}}.$$

Differentiating its two sides, we obtain

$$\frac{x^3 + 2x^2 + 3x + 4}{\sqrt{x^2 + 2x + 2}} = (2b_0x + b_1)\sqrt{x^2 + 2x + 2}$$
$$+ (b_0x^2 + b_1x + b_2) \cdot \frac{x + 1}{\sqrt{x^2 + 2x + 2}} + \lambda \cdot \frac{1}{\sqrt{x^2 + 2x + 2}}.$$

Next we clear the equation of fractions:

$$x^3 + 2x^2 + 3x + 4 = (2b_0x + b_1)(x^2 + 2x + 2) + (b_0x^2 + b_1x + b_2)(x + 1) + \lambda,$$

or

$$x^3 + 2x^2 + 3x + 4 = 3b_0x^3 + (5b_0 + 2b_1)x^2$$
$$+ (4b_0 + 3b_1 + b_2)x + (2b_1 + b_2 + \lambda).$$

Comparing the coefficients in like powers of x, we get

$$\begin{cases} 3b_0 = 1, \\ 5b_0 + 2b_1 = 2, \\ 4b_0 + 3b_1 + b_2 = 3, \\ 2b_1 + b_2 + \lambda = 4. \end{cases}$$

Solving the system, we find $b_0 + 1/3$, $b_1 = 1/6$, $b_2 = 7/6$, $\lambda = 5/2$. Consequently,

$$\int \frac{x^3 + 2x^2 + 3x + 4}{\sqrt{x^2 + 2x + 2}}\, dx = \left(\frac{1}{3}x^2 + \frac{1}{6}x + \frac{7}{6}\right)\sqrt{x^2 + 2x + 2} + \frac{5}{2}\ln|x + 1 + \sqrt{x^2 + 2x + 2}| + C.$$

Find the following integrals:

1427. $\displaystyle\int \frac{dx}{\sqrt{1 - 2x} - \sqrt[4]{1 - 2x}}$. 1428. $\displaystyle\int \frac{\sqrt[6]{x}}{1 + \sqrt[3]{x}}\, dx$.

1429. $\displaystyle\int \frac{dx}{\sqrt{x^2 - x - 1}}$.

1430. $\displaystyle\int \frac{dx}{\sqrt{-x^2 - 2x + 8}}$.

1431. $\displaystyle\int \frac{5x + 3}{\sqrt{-x^2 + 4x + 5}}\, dx$. 1432. $\displaystyle\int \frac{3x + 2}{\sqrt{x^2 + x + 2}}\, dx$.

1433. $\displaystyle\int \frac{dx}{(x + 2)\sqrt{x^2 + 2x}}$. 1434. $\displaystyle\int \frac{dx}{x\sqrt{2x^2 - 2x - 1}}$.

1435. $\displaystyle\int \frac{x - 1}{(x + 1)\sqrt{x^2 + 1}}\, dx$. 1436. $\displaystyle\int \frac{x^2 + 2x + 3}{\sqrt{-x^2 + 4x}}\, dx$.

9.3.6. Integrals of differential binomials $\int x^m(a + bx^n)^p\, dx$ where m, n, p are rational numbers. As was proved by Chebyshev, integrals of differential binomials can be expressed in terms of elementary functions only in three cases:

(1) p is an integer; then the given integral reduces to the integral of a rational function by means of the substitution $x = t^s$, where s is the least common multiple of the denominators of the fractions m and n;

(2) $(m + 1)/n$ is an integer; in this case the given integral can be rationalized by means of the substitution $a + bx^n = t^s$;

(3) $(m + 1)/n + p$ is an integer; in this case the same aim attained by means of the substitution $ax^{-n} + b = t^s$, where s is the denominator of the fraction p.

1437. Find the integral $\displaystyle\int \frac{dx}{\sqrt{x}(\sqrt[4]{x} + 1)^{10}}$.

Solution. Here the integrand can be written as $x^{-1/2}(x^{1/4} + 1)^{-10}$, i.e. $p = -10$ is an integer. Hence, we have the first case of integrability of a differentiable binomial. Therefore, we should apply the substitution $x = t^4$; then $dx = 4t^3dt$, and the desired integral assumes the form

$$\int \frac{dx}{\sqrt{x}(\sqrt[4]{x}+1)^{10}} = \int \frac{4t^3dt}{t^2(t+1)^{10}} = 4\int \frac{t\,dt}{(t+1)^{10}}.$$

The last integral can be found as follows:

$$\int \frac{t\,dt}{(t+1)^{10}} = \int \frac{t+1-1}{(t+1)^{10}}dt = \int \frac{dt}{(t+1)^9} - \int \frac{dt}{(t+1)^{10}}$$

$$= \int (t+1)^{-9}d(t+1) - \int (t+1)^{-10}d(t+1) = -\frac{1}{8(t+1)^8} + \frac{1}{9(t+1)^9} + C.$$

Thus

$$\int \frac{dx}{\sqrt{x}(\sqrt[4]{x}+1)^{10}} = -\frac{1}{2(\sqrt[4]{x}+1)^8} + \frac{4}{9(\sqrt[4]{x}+1)^9} + C.$$

1438. Find the integral $\displaystyle\int \frac{x^3dx}{(a^2 - x^2)\sqrt{a^2 - x^2}}$.

Solution. Having rewritten the integrand in the form $x^3(a^2 - x^2)^{-3/2}$, we get $m = 3$, $n = 2$, $p = -3/2$. Since $(m + 1)/n = (3 + 1)/2 = 2$ is an integer, we have the second case of integrability. Using the substitution $a^2 - x^2 = t^2$, we obtain $-2xdx = 2t\,dt$, $xdx = -t\,dt$, $x^2 = a^2 - t^2$. Consequently,

$$\int x^3(a^2 - x^2)^{-3/2}dx = -\int (a^2 - t^2)t^{-3} \cdot t\,dt$$

$$= -\int \frac{a^2 - t^2}{t^2}dt = \int dt - a^2\int \frac{dt}{t^2} = t + \frac{a^2}{t^2} + C$$

$$= \frac{t^2 + a^2}{t} + C = \frac{2a^2 - x^2}{\sqrt{a^2 - x^2}} + C.$$

1439. Find the integral $\displaystyle\int \frac{dx}{x^4\sqrt{1 + x^2}}$.

Solution. Here $m = -4$, $n = 2$, $p = -1/2$ and $(m + 1)/n + p = (-4 + 1)/2 - 1/2 = -2$ is an integer. Therefore, we have here the third case of integrability of a differential binomial. We put $x^{-2} + 1 = t^2$; then $-2x^{-3}dx = 2t\,dt$, $x^{-3}dx = -t\,dt$. We transform the given integral as follows:

$$I = \int \frac{dx}{x^4\sqrt{1 + x^2}} = \int x^{-4}(1 + x^2)^{-1/2}dx$$

$$= \int x^{-4}[x^2(x^{-2}+1)]^{-1/2}dx = \int x^{-2}(x^{-2}+1)^{-1/2}x^{-3}dx.$$

Consequently,

$$I = -\int (t^2-1)t^{-1}\cdot t\,dt = -\int (t^2-1)dt$$

$$= t - \frac{t^3}{3} + C = \sqrt{x^{-2}+1} - \frac{\sqrt{(x^{-2}+1)^3}}{3} + C$$

$$= \frac{\sqrt{1+x^2}}{x} - \frac{\sqrt{(1+x^2)^3}}{3x^3} + C = \frac{(2x^2-1)\sqrt{1+x^2}}{3x^3} + C.$$

Find the following integrals:

1440. $\displaystyle\int \frac{dx}{x(\sqrt[3]{x}+1)^2}$. **1441.** $\displaystyle\int \frac{\sqrt{x}}{\sqrt{\sqrt{x}+1}}\,dx$.

1442. $\displaystyle\int \frac{dx}{\sqrt[3]{1+x^3}}$. **1443.** $\displaystyle\int \frac{dx}{x\sqrt{1+x^3}}$.

1444. $\displaystyle\int \sqrt[3]{x}\sqrt{5x\sqrt[3]{x}+3}\,dx$. **1445.** $\displaystyle\int \frac{dx}{x^3\sqrt[3]{2-x^3}}$.

9.4. Integration of Trigonometric Functions

9.4.1. Integrals of the form $\int R(\sin x, \cos x)dx$, where R is a rational function. Integrals of the indicated form can be reduced to integrals of rational functions with the aid of the so-called *universal trigonometric substitution* $\tan(x/2) = t$.

As a result of this substitution we have

$$\sin x = \frac{2\tan(x/2)}{1+\tan^2(x/2)} = \frac{2t}{1+t^2}; \quad \cos x = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} = \frac{1-t^2}{1+t^2};$$

$$x = 2\arctan t; \quad dt = \frac{2dt}{1+t^2}.$$

1446. Find the integral $\displaystyle\int \frac{dx}{4\sin x + 3\cos x + 5}$.

Solution. The integrand depends rationally on $\sin x$ and $\cos x$. Using the substitution $\tan(x/2) = t$, we get $\sin x = \dfrac{2t}{1+t^2}$, $\cos x = \dfrac{1-t^2}{1+t^2}$, $dx = \dfrac{2dt}{1+t^2}$ and

$$\int \frac{dx}{4\sin x + 3\cos x + 5} = \int \frac{\dfrac{2dt}{1+t^2}}{4 \cdot \dfrac{2t}{1+t^2} + 3 \cdot \dfrac{1-t^2}{1+t^2} + 5}$$

$$= 2\int \frac{dt}{2t^2 + 8t + 8} = \int \frac{dt}{(t+2)^2} = -\frac{1}{t+2} + C.$$

Returning to the old variable, we obtain

$$\int \frac{dx}{4\sin x + 3\cos x + 5} = -\frac{1}{\tan(x/2) + 2} + C.$$

1447. Find the integral $\displaystyle\int \frac{dx}{(a^2+b^2) - (a^2-b^2)\cos x}$.

Solution. Setting $\tan(x/2) = t$, we get

$$\int \frac{dx}{(a^2+b^2)-(a^2-b^2)\cos x} = \int \frac{\dfrac{2dt}{1+t^2}}{(a^2+b^2)-(a^2-b^2)\cdot \dfrac{1-t^2}{1+t^2}}$$

$$= 2\int \frac{dt}{(a^2+b^2)(1+t^2) - (a^2-b^2)(1-t^2)} = \int \frac{dt}{a^2t^2 + b^2}$$

$$= \frac{1}{a}\int \frac{d(at)}{(at)^2+b^2} = \frac{1}{ab}\arctan\frac{at}{b} + C = \frac{1}{ab}\arctan\left(\frac{a}{b}\cdot\tan\frac{x}{2}\right) + C.$$

In many cases the universal substitution $\tan(x/2) = t$ leads to cumbersome calculations because when it is applied $\sin x$ and $\cos x$ are expressed in terms of t as rational fractions containing t^2.

In some special cases, calculation of the integrals of the form $\int R(\sin x, \cos x)dx$ can be simplified.

1. If $R(\sin x, \cos x)$ is an odd function with respect to $\sin x$, that is, if $R(-\sin x, \cos x) = -R(\sin x, \cos x)$, the integral can be rationalized by means of the substitution $\cos x = t$.

2. If $R(\sin x, \cos x)$ is an odd function with respect to $\cos x$, that is, if $R(\sin x, -\cos x) = -R(\sin x, \cos x)$, then the integral can be rationalized by means of the substitution $\sin x = t$.

3. If $R(\sin x, \cos x)$ is an even functions with respect to $\sin x$ and $\cos x$, that is, if $R(-\sin x, -\cos x) = R(\sin x, \cos x)$, then the purpose can be achieved by means of the substitution $\tan x = t$.

1448. Find the integral $\int \frac{(\sin x + \sin^3 x)dx}{\cos 2x}$.

Solution. Since the integrand is odd with respect to the sine, we put $\cos x = t$. Hence $\sin^2 x = 1 - t^2$, $\cos 2x = 2\cos^2 x - 1 = 2t^2 - 1$, $dt = -\sin x dx$. Thus we have

$$\int \frac{(\sin x + \sin^3 x)dx}{\cos 2x} = \int \frac{(2 - t^2)(-dt)}{2t^2 - 1} = \int \frac{(t^2 - 2)dt}{2t^2 - 1}$$

$$= \frac{1}{2} \int \frac{2t^2 - 4}{2t^2 - 1} dt = \frac{1}{2} \int dt - \frac{3}{2} \int \frac{dt}{2t^2 - 1}$$

$$= \frac{t}{2} - \frac{3}{2\sqrt{2}} \int \frac{d(t\sqrt{2})}{2t^2 - 1} = \frac{t}{2} - \frac{3}{2\sqrt{2}} \ln \left| \frac{t\sqrt{2} - 1}{t\sqrt{2} + 1} \right| + C.$$

And consequently,

$$\int \frac{(\sin x + \sin^3 x)dx}{\cos 2x} = \frac{1}{2} \cos x - \frac{3}{2\sqrt{2}} \ln \left| \frac{\sqrt{2}\cos x - 1}{\sqrt{2}\cos x + 1} \right| + C.$$

Note that in the present case the integral can always be written in the form $\int R^*(\sin^2 x, \cos x)\sin x dx$.

1449. Find the integral $\int \frac{(\cos^3 x + \cos^5 x)dx}{\sin^2 x + \sin^4 x}$.

Solution. Here the integrand is odd with respect to the cosine. Therefore, we apply the substitution $\sin x = t$; then $\cos^2 x = 1 - \sin^2 x = 1 - t^2$, $\cos x dx = dt$. Consequently,

$$\int \frac{(\cos^3 x + \cos^5 x)dx}{\sin^2 x + \sin^4 x} = \int \frac{\cos^2 x(1 + \cos^2 x)\cos x dx}{\sin^2 x + \sin^4 x} = \int \frac{(1 - t^2)(2 - t^2)dt}{t^2 + t^4}.$$

But

$$\frac{(1 - t^2)(2 - t^2)}{t^2(1 + t^2)} = 1 + \frac{2}{t^2} - \frac{6}{1 + t^2}$$

and, hence,

$$\int \frac{(1 - t^2)(2 - t^2)dt}{t^2(1 + t^2)} = t - \frac{2}{t} - 6 \arctan t + C.$$

We finally obtain

$$\int \frac{(\cos^3 x + \cos^5 x)dx}{\sin^2 x + \sin^4 x} = \sin x - \frac{2}{\sin x} - 6 \arctan(\sin x) + C.$$

Note that in the case being considered the integral can always be written in the form $\int R^*(\sin x, \cos^2 x)\cos x dx$.

1450. Find the integral $\displaystyle\int \frac{dx}{\sin^2 x + 2\sin x \cos x - \cos^2 x}$.

Solution. The integrand is even with respect to the sine and cosine. Putting $\tan x = t$, we get

$$\sin x = \frac{\tan x}{\sqrt{1+\tan^2 x}} = \frac{1}{\sqrt{1+t^2}}; \quad \cos x = \frac{1}{\sqrt{1+\tan^2 x}} = \frac{1}{\sqrt{1+t^2}};$$

$$x = \arctan t; \quad dx = \frac{dt}{1+t^2}.$$

From this we obtain

$$\int \frac{dx}{\sin^2 x + 2\sin x \cos x - \cos^2 x}$$

$$= \int \frac{\frac{dt}{1+t^2}}{\frac{t^2}{1+t^2} + \frac{2t}{\sqrt{1+t^2}} \cdot \frac{1}{\sqrt{1+t^2}} - \frac{1}{1+t^2}} = \int \frac{dt}{t^2 + 2t - 1},$$

but

$$\int \frac{dt}{t^2+2t-1} = \int \frac{d(t+1)}{(t+1)^2 - (\sqrt{2})^2} = \frac{1}{2\sqrt{2}} \ln \left| \frac{t+1-\sqrt{2}}{t+1+\sqrt{2}} \right| + C.$$

Therefore we have

$$\int \frac{dx}{\sin^2 x + 2\sin x \cos x - \cos^2 x} = \frac{1}{2\sqrt{2}} \ln \left| \frac{\tan x + 1 - \sqrt{2}}{\tan x + 1 + \sqrt{2}} \right| + C.$$

Note that the evaluation of the integral can be simplified by dividing the numerator and the denominator of the original integral by $\cos^2 x$:

$$\int \frac{dx}{\sin^2 x + 2\sin x \cos x - \cos^2 x}$$

$$= \int \frac{\frac{dx}{\cos^2 x}}{\tan^2 x + 2\tan x - 1} = \int \frac{d(\tan x)}{\tan^2 x + 2\tan x - 1}.$$

9.4.2. Integrals of the form $\int \sin^m x \cos^n x\, dx$. Here we shall consider two cases which are of especial significance.

Case 1. At least one of the exponents m and n is an odd positive number.

If n is an odd positive number, then we apply the substitution $\sin x = t$; now if m is an odd positive number, the substitution $\cos x = t$.

1451. Find the integral $\int \sin^4 x \cos^5 x\, dx$.

Solution. Setting $\sin x = t$, $\cos x\, dx = dt$, we get

$$\int \sin^4 x \cos^5 x\, dx = \int \sin^4 x (1 - \sin^2 x)^2 \cos x dx = \int t^4 (1 - t^2)^2 dt$$

$$= \int t^4 dt - 2\int t^6 dt + \int t^8 dt = \frac{1}{5} t^5 - \frac{2}{7} t^7 + \frac{1}{9} t^9 + C$$

$$= \frac{1}{5} \sin^5 x - \frac{2}{7} \sin^7 x + \frac{1}{9} \sin^9 x + C.$$

1452. Find the integral $\int \frac{\sin^3 x\, dx}{\cos x \sqrt[3]{\cos x}}$.

Solution. We have

$$\int \frac{\sin^3 x}{\cos x \sqrt[3]{\cos x}} dx = \int \sin^3 x \cdot \cos^{-4/3} x dx = \int (1 - \cos^2 x) \cos^{-4/3} x \sin x dx.$$

Putting $\cos x = t$, $-\sin x\, dx = dt$, we get

$$\int \frac{\sin^3 x}{\cos x \sqrt[3]{\cos x}} dx = - \int (1 - t^2) t^{-4/3} dt = - \int t^{-4/3} dt + \int t^{2/3} dt$$

$$= 3t^{-1/3} + \frac{3}{5} t^{5/3} + C = \frac{3}{\sqrt[3]{\cos x}} + \frac{3}{5} \cos x \cdot \sqrt[3]{\cos^2 x} + C.$$

Case 2. Both exponents m and n are even positive numbers. Here the integrand must be transformed with the aid of the formulas

$$\sin x \cos x = \frac{1}{2} \sin 2x, \qquad (1)$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x), \qquad (2)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x). \qquad (3)$$

1453. Find the integral $\int \sin^2 x \cos^2 x \, dx$.
Solution. It follows from formula (1) that

$$\sin^2 x \cos^2 x = (\sin x \cos x)^2 = \left(\frac{1}{2}\sin 2x\right)^2 = \frac{1}{4}\sin^2 2x.$$

Applying now formula (2), we get

$$\sin^2 x \cos^2 x = \frac{1}{4} \cdot \frac{1-\cos 4x}{2} = \frac{1}{8}(1-\cos 4x).$$

Thus we have

$$\int \sin^2 x \cos^2 x \, dx = \frac{1}{8}\int (1-\cos 4x)dx = \frac{1}{8}\int dx - \frac{1}{8}\int \cos 4x dx$$
$$= \frac{1}{8}x - \frac{1}{32}\sin 4x + C.$$

1454. Find the integral $\int \cos^6 x \, dx$.
Solution. Using formula (3), we obtain

$$\int \cos^6 x \, dx = \int (\cos^2 x)^3 \, dx = \int \left(\frac{1+\cos 2x}{2}\right)^3 dx$$
$$= \frac{1}{8}\int (1 + 3\cos 2x + 3\cos^2 2x + \cos^3 2x)\, dx$$
$$= \frac{1}{8}\int dx + \frac{3}{8}\int \cos 2x \, dx + \frac{3}{8}\int \cos^2 2x \, dx + \frac{1}{8}\int \cos^3 2x \, dx$$
$$= \frac{1}{8}x + \frac{3}{16}\sin 2x + \frac{3}{8}\int \frac{1+\cos 4x}{2} dx$$
$$+ \frac{1}{8}\int (1-\sin^2 2x)\cos 2x \, dx = \frac{1}{8}x + \frac{3}{16}\sin 2x$$
$$+ \frac{3}{16}\int dx + \frac{3}{16}\int \cos 4x \, dx + \frac{1}{8}\int \cos 2x \, dx$$
$$- \frac{1}{8}\int \sin^2 2x \cdot \frac{1}{2}\, d(\sin 2x) = \frac{1}{8}x + \frac{3}{16}\sin 2x$$
$$+ \frac{3}{16}x + \frac{3}{64}\sin 4x + \frac{1}{16}\sin 2x - \frac{1}{48}\sin^3 2x + C$$

$$= \frac{5}{16}x + \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C.$$

1455. Find the integral $\int \sin^2 x \cos^4 x\, dx$.

Solution. We have

$$\int \sin^2 x \cos^4 x\, dx = \int (\sin x \cos x)^2 \cos^2 x\, dx = \int \left(\frac{1}{2} \sin 2x\right)^2$$

$$\times \frac{1 + \cos 2x}{2} dx = \frac{1}{8} \int \sin^2 2x\, dx + \frac{1}{8} \int \sin^2 2x \cos 2x\, dx$$

$$= \frac{1}{8} \int \frac{1 - \cos 4x}{2} dx + \frac{1}{8} \int \sin^2 2x \cdot \frac{1}{2}\, d(\sin 2x)$$

$$= \frac{1}{16} \int dx - \frac{1}{16} \int \cos 4x\, dx + \frac{1}{16} \int \sin^2 2x\, d(\sin 2x)$$

$$= \frac{1}{16} x - \frac{1}{64} \sin 4x + \frac{1}{48} \sin^3 2x + C.$$

9.4.3. Integrals of the form $\int \tan^m x\, dx$ and $\int \cot^m x\, dx$, where m is a positive integer. Evaluation of such integrals can be carried out with the aid of the formula

$$\tan^2 x = \sec^2 x - 1 \text{ (or } \cot^2 x = \operatorname{cosec}^2 x - 1),$$

by means of which the degree of the tangent or cotangent is consecutively lowered.

1456. Find the integral $\int \tan^7 x\, dx$.

Solution. We have

$$\int \tan^7 x\, dx = \int \tan^5 x (\sec^2 x - 1)\, dx = \int \tan^5 x\, d(\tan x) - \int \tan^5 x\, dx =$$

$$= \frac{\tan^6 x}{6} - \int \tan^3 x (\sec^2 x - 1)\, dx = \frac{\tan^6 x}{6} - \frac{\tan^4 x}{4} + \int \tan x (\sec^2 x - 1)\, dx$$

$$= \frac{\tan^6 x}{6} - \frac{\tan^4 x}{4} + \frac{\tan^2 x}{2} + \ln|\cos x| + C.$$

1457. Find the integral $\int \cot^6 x\, dx$.

Solution. We have

$$\int \cot^6 x\,dx = \int \cot^4 x(\operatorname{cosec}^2 x - 1)\,dx = -\int \cot^4 x\,d(\cot x)$$

$$-\int \cot^4 x\,dx = -\frac{\cot^5 x}{5} - \int \cot^2 x(\operatorname{cosec}^2 x - 1)\,dx$$

$$= -\frac{\cot^5 x}{5} + \frac{\cot^3 x}{3}\int(\operatorname{cosec}^2 x - 1)\,dx = -\frac{\cot^5 x}{5} + \frac{\cot^3 x}{3} - \cot x - x + C.$$

9.4.4. Integrals of the form $\int \tan^m x \sec^n x\,dx$ and $\int \cot^m x \operatorname{cosec}^n x\,dx$, where n is a positive integer. These integrals can be found by analogy with the integrals discussed in 9.4.3 with the aid of the formula

$$\sec^2 x = 1 + \tan^2 x \ (\text{or } \operatorname{cosec}^2 x = 1 + \cot^2 x).$$

1458. Find the integral $\int \tan^4 x \sec^6 x\,dx$.

Solution. We have

$$\int \tan^4 x \sec^6 x\,dx = \int \tan^4 x \sec^4 x \sec^2 x\,dx = \int \tan^4 x$$

$$\times (1 + \tan^2 x)^2 d(\tan x) = \int \tan^4 x\,d(\tan x) + 2\int \tan^6 x\,d(\tan x)$$

$$+ \int \tan^8 x\,d(\tan x) = \frac{1}{5}\tan^5 x + \frac{2}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C.$$

1459. Find the integral $\int \frac{dx}{\sin^4 x}$.

Solution. We have

$$\int \frac{dx}{\sin^4 x} = \int \operatorname{cosec}^4 x\,dx = \int \operatorname{cosec}^2 x \cdot \operatorname{cosec}^2 x\,dx$$

$$= \int (1 + \cot^2 x)\operatorname{cosec}^2 x\,dx = \int \operatorname{cosec}^2 x\,dx - \int \cot^2 x\,d(\cot x)$$

$$= -\cot x - \frac{1}{3}\cot^3 x + C.$$

9.4.5. Integrals of the form $\int \sec^{2n+1} x\,dx$ and $\int \operatorname{cosec}^{2n+1} x\,dx$. Integrals of an odd positive degree of a secant or cosecant can be most easily found by the recurrence formulas

$$\int \sec^{2n+1}x\,dx = \frac{1}{2n}\cdot\frac{\sin x}{\cos^{2n}x} + \left(1-\frac{1}{2n}\right)\int \sec^{2n-1}x\,dx, \tag{1}$$

$$\int \operatorname{cosec}^{2n+1}x\,dx = -\frac{1}{2n}\cdot\frac{\cos x}{\sin^{2n}x} + \left(1-\frac{1}{2n}\right)\int \operatorname{cosec}^{2n-1}x\,dx. \tag{2}$$

1460. Find the integral $\int \operatorname{cosec}^5 x\,dx$.

Solution. Applying recurrence formula (2) for $2n+1=5$, that is, for $n=2$, we obtain

$$\int \operatorname{cosec}^5 x\,dx = -\frac{1}{4}\cdot\frac{\cos x}{\sin^4 x} + \frac{3}{4}\int \operatorname{cosec}^3 x\,dx;$$

setting now $2n+1=3$, i.e. $n=1$, we get from the same formula

$$\int \operatorname{cosec}^3 x\,dx = -\frac{1}{2}\cdot\frac{\cos x}{\sin^2 x} + \frac{1}{2}\cdot\int \operatorname{cosec} x\,dx,$$

but

$$\int \operatorname{cosec} x\,dx = \int \frac{dx}{\sin x} = \ln\left|\tan\frac{x}{2}\right| + C.$$

Consequently,

$$\int \operatorname{cosec}^3 x\,dx = -\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + C.$$

$$\int \operatorname{cosec}^5 x\,dx = -\frac{\cos x}{4\sin^4 x} - \frac{3\cos x}{8\sin^2 x} + \frac{3}{8}\ln\left|\tan\frac{x}{2}\right| + C.$$

9.4.6. Integrals of the form $\int \sin mx \cos nx\,dx$, $\int \cos mx \cos nx\,dx$, $\int \sin mx \sin nx\,dx$. The trigonometric formulas

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)], \tag{1}$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)], \tag{2}$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \tag{3}$$

make it possible to represent a product of trigonometric functions as their sum.

1461. Find the integral $\int \sin 2x \cos 5x \, dx$.

Solution. Using formula (1), we get

$$\int \sin 2x \cos 5x \, dx = \frac{1}{2} \int [\sin 7x + \sin(-3x)] \, dx$$

$$= \frac{1}{2} \int \sin 7x \, dx - \frac{1}{2} \int \sin 3x \, dx = -\frac{1}{14} \cos 7x + \frac{1}{6} \cos 3x + C.$$

1462. Find the integral $\int \cos x \cos \frac{x}{2} \cos \frac{x}{4} \, dx$.

Solution. Applying formula (2) to the product $\cos x \cos \frac{x}{2}$, we get

$$\int \cos x \cdot \cos \frac{x}{2} \cdot \cos \frac{x}{4} \, dx = \frac{1}{2} \int \left(\cos \frac{3x}{2} + \cos \frac{x}{2} \right) \cos \frac{x}{4} \, dx$$

$$= \frac{1}{2} \int \cos \frac{3x}{2} \cos \frac{x}{4} \, dx + \frac{1}{2} \int \cos \frac{x}{2} \cos \frac{x}{4} \, dx.$$

Using again the same formula, we obtain

$$\int \cos x \cos \frac{x}{2} \cos \frac{x}{4} \, dx = \frac{1}{4} \int \left(\cos \frac{7x}{4} + \cos \frac{5x}{4} \right) dx$$

$$+ \frac{1}{4} \int \left(\cos \frac{3x}{4} x + \cos \frac{x}{4} \right) dx = \frac{1}{7} \sin \frac{7x}{4} + \frac{1}{5} \sin \frac{5x}{4} + \frac{1}{3} \sin \frac{3x}{4} + \sin \frac{x}{4} + C.$$

Find the following integrals:

1463. $\int \frac{dx}{3 + 5 \sin x + 3 \cos x}$. **1464.** $\int \frac{dx}{1 - \sin x}$.

1465. $\int \frac{\cos^2 x \, dx}{\sin^2 x + 4 \sin x \cos x}$. **1466.** $\int \frac{\cos^3 x \, dx}{\sin^2 x + \sin x}$.

Hint. Put $\cot x = t$.

1467. $\int \frac{\sin 2x \, dx}{\cos^3 x - \sin^2 x - 1}$. **1468.** $\int \sin^3 x \, dx$.

1469. $\int \frac{\cos^5 x}{\sin x} dx$. **1470.** $\int \sin^2(x/4) \cos^2(x/4) \, dx$.

1471. $\int \cos^4 x\, dx$. **1472.** $\int \tan^4(x/2)\, dx$.

1473. $\int \cot^3 3x\, dx$. **1474.** $\int \sec^6 x\, dx$.

1475. $\int \frac{\cos^2 x}{\sin^4 x}\, dx$. **1476.** $\int \sec^3 x\, dx$.

1477. $\int \cot^2 x \operatorname{cosec} x\, dx$. **1478.** $\int \sin 3x \sin x\, dx$.

1479. $\int \cos(x/2)\cos(x/3)\, dx$.

9.4.7. Trigonometric substitutions. Integrals of the form $\int R\,(x\sqrt{a^2 - x^2})\, dx$, $\int R\,(x\sqrt{a^2 + x^2})\, dx$, $\int R\,(x, \sqrt{x^2 - a^2})\, dx$ can be reduced to integrals of the function rational with respect to $\sin t$ and $\cos t$ by means of the requisite trigonometric substitution:
$x = a \sin t$ (or $x = a \cos t$) for the first integral,
$x = a \tan t$ (or $x = a \cot t$) for the second integral, and
$x = a \sec t$ (or $x = a \operatorname{cosec} t$) for the third integral.

1480. Find the integral $I = \int \frac{\sqrt{a^2 - x^2}}{x}\, dx$.

Solution. We put $x = a \sin t$; then $dx = a \cos t\, dt$ and the integral in question assumes the form

$$I = \int \frac{\sqrt{a^2 - a^2 \sin^2 t}}{a \sin t}\, a \cos t\, dt = a\int \frac{\cos^2 t}{\sin t}\, dt = a\int \frac{1 - \sin^2 t}{\sin t}\, dt$$
$$= a\int \frac{dt}{\sin t} - a\int \sin t\, dt = a \ln\,|\operatorname{cosec} t - \cot t\,| + a\cos t + C.$$

To evaluate the integral $\int \frac{dt}{\sin t}$, we have used the formula $\int \frac{dt}{\sin t} =$ $= \ln|\operatorname{cosec} t - \cot t| + C$ since it makes it easier to pass to the old variable x. Thus we obtain

$$I = a \ln \left|\frac{1}{\sin t} - \frac{\cos t}{\sin t}\right| + a\cos t + C,$$

where $\sin t = x/a$, $\cos t = \sqrt{a^2 - x^2}/a$. Consequently,

$$I = a \ln\left|\frac{a - \sqrt{a^2 - x^2}}{x}\right| + \sqrt{a^2 - x^2} + C.$$

1481. Find the integral $I = \int \frac{dx}{x\sqrt{a^2 + x^2}}$.

Solution. Let us apply the substitution $x = a \tan t$, whence $dx = a \sec^2 t\, dt$. Then we have

$$I = \int \frac{a \sec^2 t\, dt}{a \tan t \sqrt{a^2 + a^2 \tan^2 t}} = \frac{1}{a} \int \frac{\sec^2 t\, dt}{\tan t \sec t}$$

$$= \frac{1}{a} \int \frac{\sec t}{\tan t} dt = \frac{1}{a} \int \frac{dt}{\sin t} = \frac{1}{a} \ln |\operatorname{cosec} t - \cot t| + C,$$

where $\tan t = x/a$ and, consequently, $\cot t = a/x$, $\operatorname{cosec} t = \sqrt{1 + \cot^2 t} = \sqrt{a^2 + x^2}/x$. Thus we obtain

$$I = \frac{1}{a} \ln\left|\frac{\sqrt{a^2 + x^2} - a}{x}\right| + C.$$

1482. Find the integral $I = \int \frac{x^2 dx}{\sqrt{x^2 - a^2}}$.

Solution. We apply the substitution $x = a \sec t$, whence $dx = a \sec t \tan t\, dt$. Then we get

$$I = \int \frac{a^2 \sec^2 t \cdot a \sec t \cdot \tan t}{\sqrt{a^2 \sec^2 t - a^2}} dt = a^2 \int \sec^3 t\, dt.$$

Next we apply recurrence formula (1) from 9.4.5 for $n = 1$:

$$\int \sec^3 t\, dt = \frac{1}{2} \frac{\sin t}{\cos^2 t} + \frac{1}{2} \int \sec t\, dt = \frac{\sin t}{2\cos^2 t} + \frac{1}{2} \int \frac{dt}{\cos t}$$

$$= \frac{\sin t}{2\cos^2 t} + \frac{1}{2} \ln |\sec t + \tan t| + C,$$

where $\sec t = x/a$, $\cos t = a/x$, $\sin t = \sqrt{x^2 - a^2}/x$, $\tan t = \sqrt{x^2 - a^2}/a$. Consequently,

$$I = \frac{a^2 \sin t}{2\cos^2 t} + \frac{a^2}{2} \ln |\sec t + \tan t| + C$$

$$= \frac{x}{2}\sqrt{x^2 - a^2} + \frac{a^2}{2}\ln\left|\frac{x + \sqrt{x^2 - a^2}}{a}\right| + C.$$

Find the integrals:

1483. $\int \frac{dx}{(1 - x^2)^{3/2}}$. **1484.** $\int \frac{dx}{(a^2 + x^2)^{3/2}}$.

1485. $\int \frac{dx}{x^3\sqrt{x^2 - 1}}$.

9.5. Integration of Various Functions

Find the following integrals:

1486. $\int \sin^2 x \sin 3x \, dx$. **1487.** $\int (2x^2 - 2x + 1)e^{-x/2}dx$.

1488. $\int \frac{\ln x}{x^2} dx$. **1489.** $\int \frac{x^2 - 2}{x^2 + 1} \cdot \arctan x \, dx$.

1490. $\int (2x^2 - 1)\cos 2x \, dx$. **1491.** $\int x \ln^2 x \, dx$.

1492. $\int \frac{2e^{2x} - e^x - 3}{e^{2x} - 2e^x - 3} dx$. **1493.** $\int \arctan\sqrt{x} \, dx$.

1494. $\int \sqrt{2^x - 1} \, dx$. **1495.** $\int \frac{dx}{\sqrt[4]{1 + x^4}}$.

1496. $\int \sqrt{6 + 4x - 2x^2} \, dx$. **1497.** $\int e^{2x} \sin e^x \, dx$.

1498. $\int \frac{dx}{\cos^2 x\sqrt{2 + 5\tan^2 x}}$. **1499.** $\int \sin 2x \ln \cos x \, dx$.

1500. $\int (x + 2)\cos(x^2 + 4x + 1) \, dx$.

1501. $\int \frac{x \cos x}{\sin^3 x} dx$. **1502.** $\int \frac{xe^x}{\sqrt{1 + e^x}} dx$.

1503. $\int \ln(x^2 + x)\,dx$. **1504.** $\int \dfrac{dx}{x^4 + x^2}$.

1505. $\int \cos \ln x\,dx$. **1506.** $\int \dfrac{1 + \sqrt[6]{x}}{(\sqrt[3]{x} - \sqrt[4]{x})\sqrt[4]{x^3}}\,dx$.

1507. $\int e^{\alpha x} \sin \beta x\,dx$. **1508.** $\int e^{\alpha x} \cos \beta x\,dx$.

1509. $\int \dfrac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$. **1510.** $\int \dfrac{dx}{\sin^2 x \cos^2 x}$.

1511. $\int \dfrac{dx}{(1 + x^2)^2}$.

Chapter 10

Definite Integrals

10.1. Calculation of Definite Integral

Suppose the function $f(x)$ is defined on the closed interval $[a, b]$. Let us divide the interval $[a, b]$ into n arbitrary parts by the points $a = x_0 < x_1 < < x_2 < \dots < x_{n-1} < x_n = b$, choose an arbitrary point ξ_k in each subinterval $[x_{k-1}, x_k]$ and find the length of every such subinterval: $\Delta x_k = x_k - x_{k-1}$.

The *integral sum* for the function $f(x)$ on the closed interval $[a, b]$ is the sum of the form

$$\sum_{k=1}^{n} f(\xi_k)\Delta x_k = f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + \dots + f(\xi_n)\Delta x_n.$$

The *definite integral* of the function $f(x)$ on the closed interval $[a, b]$ (or in the limit from a to b) is the limit of the integral sum under the condition that the length of the greatest subinterval tends to zero:

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(\xi_k)\Delta x_k.$$

If the function $f(x)$ is continuous on the interval $[a, b]$, then the limit of the integral sum exists and does not depend on the method of partitioning of the interval $[a, b]$ into subintervals and on the choice of the points ξ_k **(theorem on the existence of a definite integral).**

If $f(x) > 0$ on $[a, b]$, then, in geometrical interpretation, the definite integral $\int_a^b f(x)\,dx$ is the area of a *curvilinear trapezoid*, a figure bounded by the lines $y = f(x)$, $x = a$, $x = b$, $y = 0$ (Fig. 42).

Basic properties of a definite integral

$1°.\ \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$ $\qquad$ $2°.\ \int_a^a f(x)\,dx = 0.$

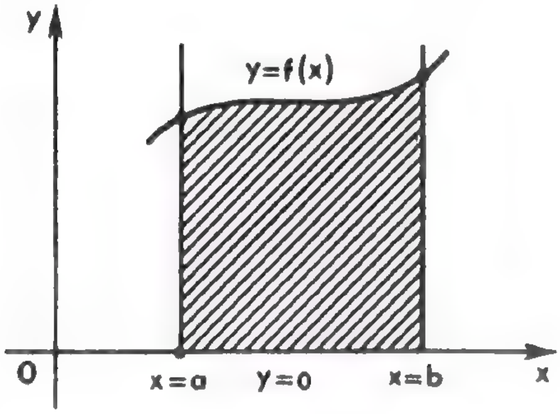

Fig. 42

$$3°.\ \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

$$4°.\ \int_a^b [f_1(x) \pm f_2(x)]\,dx = \int_a^b f_1(x)\,dx \pm \int_a^b f_2(x)\,dx.$$

$$5°.\ \int_a^b C \cdot f(x)\,dx = C \cdot \int_a^b f(x)\,dx,\text{ where } C \text{ is a constant.}$$

6°. *Evaluation of a definite integral:* if $m \leqslant f(x) \leqslant M$ on $[a, b]$, then

$$m(b - a) < \int_a^b f(x)\,dx < M(b - a).$$

Rules of computing definite integrals

1. *Newton-Leibniz' formula:*

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a),$$

where $F(x)$ is an antiderivative of $f(x)$, i.e. $F'(x) = f(x)$.

2. *Integration by parts:*

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

where $u = u(x)$, $v = v(x)$ are continuously differentiable functions on the interval $[a, b]$.

3. *Change of variable:*

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)\,dt,$$

where $x = \varphi(t)$ is a function continuous together with its derivative $\varphi'(t)$ on the interval $\alpha \leqslant t \leqslant \beta$, $a = \varphi(\alpha)$, $b = \varphi(\beta)$, and $f[\varphi(t)]$ is a function continuous on $[\alpha, \beta]$.

4. If $f(x)$ is an odd function, i.e. $f(-x) = -f(x)$, then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

If $f(x)$ is an even function, i.e. $f(-x) = f(x)$, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$$

1512. Evaluate the integral $\int_0^1 x^2\,dx$ as a limit of an integral sum.

Solution. Here $f(x) = x^2$, $a = 0$, $b = 1$. We divide the interval $[0, 1]$ into n congruent parts and obtain $\Delta x_k = (b - a)/n = 1/n$. Next we choose $\xi_k = x_k$ and have

$$x_0 = 0, x_1 = \frac{1}{n}, x_2 = \frac{2}{n}, \ldots, x_{n-1} = \frac{n-1}{n}, x_n = \frac{n}{n} = 1;$$

$$f(\xi_1) = \left(\frac{1}{n}\right)^2; f(\xi_2) = \left(\frac{2}{n}\right)^2, \ldots, f(\xi_k) = \left(\frac{n}{n}\right)^2; f(\xi_k)\Delta x_k = \left(\frac{k}{n}\right)^2 \times \frac{1}{n}.$$

Consequently,

$$\int_0^1 x^2\,dx = \lim_{n\to\infty} \frac{1^2 + 2^2 + 3^2 + \ldots + n^2}{n^3} = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3}$$

$$= \lim_{n\to\infty} \frac{\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)}{6} = \frac{1}{3}.$$

We have used here the formula for the sum of the squares of natural numbers.

1513. Evaluate $\int\limits_{\pi/6}^{\pi/4} \frac{dx}{\cos^2 x}$ by the Newton-Leibniz formula.

Solution. We have

$$\int\limits_{\pi/6}^{\pi/4} \frac{dx}{\cos^2 x} = \tan x \Big|_{\pi/6}^{\pi/4} = \tan\frac{\pi}{4} - \tan\frac{\pi}{6} = 1 - \frac{\sqrt{3}}{3}.$$

1514. Evaluate the integral $\int\limits_{10}^{18} \frac{\cos x\, dx}{\sqrt{1 + x^4}}$.

Solution. Since $\cos x \leqslant 1$, the condition $x > 10$ yields the inequality $\left|\frac{\cos x}{\sqrt{1 + x^4}}\right| < 10^{-2}$. It follows that

$$\left|\int\limits_{10}^{18} \frac{\cos x}{\sqrt{1 + x^4}} dx\right| < 8 \cdot 10^{-2} < 10^{-1}, \text{ i.e.} \quad \left|\int\limits_{10}^{18} \frac{\cos x}{\sqrt{1 + x^4}} dx\right| < 0.1.$$

1515. Evaluate the integral $\int\limits_{0}^{\pi/2} \frac{dx}{5 + 3\cos^2 x}$.

Solution. Since $0 \leqslant \cos^2 x \leqslant 1$, we have

$$\frac{1}{8} \leqslant \frac{1}{5 + 3\cos^2 x} \leqslant \frac{1}{5} \text{ and } \frac{\pi}{16} \leqslant \int\limits_{0}^{\pi/2} \frac{dx}{5 + 3\cos^2 x} \leqslant \frac{\pi}{10}.$$

1516. Calculate $\int\limits_{0}^{1} xe^{-x}\, dx$.

Solution. We use here the method of integration by parts. We set $u = x$, $dv = e^{-x}\, dx$, whence $du = dx$, $v = -e^{-x}$. Then

$$\int\limits_{0}^{1} xe^{-x}\, dx = -xe^{-x}\Big|_{0}^{1} + \int\limits_{0}^{1} e^{-x} dx = -e^{-1} - e^{-x}\Big|_{0}^{1} = -2e^{-1} + 1 = \frac{e - 2}{e}.$$

1517. Evaluate $\int\limits_{0}^{e} \frac{\ln^2 x}{x} dx$.

Solution. We set $\ln x = t$; then $\frac{dx}{x} = dt$; if $x = 1$, then $t = 0$; if $x = e$, then $t = 1$. Consequently,

$$\int_1^e \frac{\ln^2 x}{x}\,dx = \int_0^1 t^2 dt = \frac{1}{3}\,t^3\Big|_0^1 = \frac{1}{3}\,(1^3 - 0^3) = \frac{1}{3}\,.$$

1518. Evaluate $\int\limits_0^r \sqrt{r^2 - x^2}\,dx$.

Solution. We set $x = r\sin t$; then $dx = r\cos t\,dt$; if $x = 0$, then $t = 0$; if $x = r$, then $t = \pi/2$. Therefore,

$$\int_0^r \sqrt{r^2 - x^2}\,dx = \int_0^{\pi/2} \sqrt{r^2 - r^2\sin^2 t}\; r\cos t\,dt = r^2 \int_0^{\pi/2} \cos^2 t\,dt$$

$$= \frac{1}{2}\,r^2 \int_0^{\pi/2} (1 + \cos 2t)\,dt = \frac{1}{2}\,r^2\left[t + \frac{1}{2}\,\sin 2t\right]_0^{\pi/2}$$

$$= \frac{r^2}{2}\left[\left(\frac{\pi}{2} + \frac{1}{2}\,\sin\pi\right) - \left(0 + \frac{1}{2}\,\sin 0\right)\right] = \frac{\pi r^2}{4}\,.$$

1519. Evaluate $I = \int\limits_{-\pi/3}^{\pi/3} \frac{x\sin x}{\cos^2 x}\,dx$.

Solution. The integrand is even, therefore $I = 2\int\limits_0^{\pi/3} \frac{x\sin x}{\cos^2 x}\,dx$. We perform integration by parts, putting $u = x$, $dv = \frac{\sin x\,dx}{\cos^2 x}$; then we have $du = dx$, $v = \frac{1}{\cos x}$. From this we find

$$\int_0^{\pi/3} \frac{x\sin x}{\cos^2 x}\,dx = \frac{x}{\cos x}\Big|_0^{\pi/3} - \int_0^{\pi/3} \frac{dx}{\cos x} = \frac{\pi}{3\cos(\pi/3)} - \ln\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\Big|_0^{\pi/3}$$

$$= \frac{2\pi}{3} - \ln\tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right) + \ln\tan\frac{\pi}{4} = \frac{2\pi}{3} - \ln\tan\frac{5\pi}{12}\,.$$

Consequently,

$$I = 2\left(\frac{2\pi}{3} - \ln\tan\frac{5\pi}{12}\right).$$

1520. Calculate $I = \int\limits_{-1}^{1} \frac{x^2\arcsin x}{\sqrt{1 + x^2}}\,dx$.

Solution. The integrand is odd, consequently $I = 0$.

1521. Calculate $\int_0^1 x\,dx$ as a limit of an integral sum.

1522. Calculate $\int_0^1 e^x dx$ as a limit of an integral sum.

1523. Evaluate the integral $\int_0^1 x(1 - x)^2 dx$.

1524. Evaluate the integral $\int_0^{\pi/2} e^{\sin^2 x} dx$.

1525. Evaluate the integral $\int_{\pi/2}^{\pi} \frac{\sin x}{x} dx$.

Compute the following integrals:

1526. $\int_1^3 x^3\sqrt{x^2 - 1}\,dx$. **1527.** $\int_0^1 \frac{x\,dx}{1 + x^4}$.

1528. $\int_1^2 \frac{e^{1/x}}{x^2} dx$. **1529.** $\int_0^1 e^{x + e^x} dx$.

1530. $\int_1^{e^{\pi/2}} \cos \ln x\,dx$. **1531.** $\int_0^{\pi/6} \frac{\sin^2 x}{\cos x} dx$.

1532. $\int_{\ln 2}^{2 \ln 2} \frac{dx}{e^x - 1}$. **1533.** $\int_0^{2\pi} \cos 5x \cos x\,dx$.

1534. $\int_0^{\pi/3} \cos^3 x \sin 2x\,dx$. **1535.** $\int_0^{\pi/4} \frac{x + \sin x}{1 + \cos x} dx$.

1536. $\int_1^2 \frac{dx}{x^2 + x}$. **1537.** $\int_0^{\pi/2} e^x \cos x\,dx$.

1538. $\int_{-3}^{3} \frac{x^2 \sin 2x}{x^2 + 1} dx$.

Hint. Use the property of an odd function.

1539. $\int_{-1}^{1} x \arctan x\,dx$.

Hint. Use the property of an even function.

1540. Prove that

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if} \quad m \neq n; \\ \pi & \text{if} \quad m = n. \end{cases}$$

(m and n being positive integers).

10.2. Improper Integrals

10.2.1. Basic Notions. *Improper integrals* are: (1) integrals with infinite limits of integration; (2) integrals of unbounded functions.

The improper integral of the function $f(x)$ in the limit from a to $+\infty$ is specified by the equation

$$\int_a^{+\infty} f(x)dx = \lim_{b \to +\infty} \int_a^b f(x)dx.$$

If the limit exists and is finite, the improper integral is called *convergent*; now if the limit does not exist or is equal to infinity, it is called *divergent*.

Similarly,

$$\int_{-\infty}^{b} f(x)\, dx = \lim_{a \to -\infty} \int_a^b f(x)\, dx$$

and

$$\int_{-\infty}^{+\infty} f(x)\, dx = \lim_{\substack{a \to -\infty \\ b \to +\infty}} \int_a^b f(x)\, dx.$$

If the function $f(x)$ possesses an infinite discontinuity at a point c of the closed interval $[a, b]$ and is continuous at $a \leqslant x < c$ and $c < x \leqslant b$, then, by definition,

$$\int_a^b f(x)\, dx = \lim_{\alpha \to 0} \int_a^{c-\alpha} f(x)\, dx + \lim_{\beta \to 0} \int_{c+b}^{b} f(x)\, dx.$$

The improper integral $\int_a^b f(x)\, dx$ (where $f(c) = \infty$, $a < c < b$) is said to be *convergent*, if both limits exist on the right-hand side of the equality, and *divergent*, if either of them does not exist.

1541. Compute the improper integral $\int_0^{+\infty} \cos x\, dx$ (or establish its divergence).

Solution. We have

$$\lim_{b \to +\infty} \int_0^b \cos x \, dx = \lim_{b \to +\infty} \sin x \Big|_0^b = \lim_{b \to +\infty} (\sin b - \sin 0) = \lim_{b \to +\infty} \sin b,$$

that is, the limit does not exist. Consequently, the improper integral diverges.

1542. Compute $\int_{-\infty}^{-1} \frac{dx}{x^2}$.

Solution. We find

$$\lim_{a \to -\infty} \int_a^{-1} \frac{dx}{x^2} = \lim_{a \to -\infty} \left[-\frac{1}{x} \right]_a^{-1} = \lim_{a \to -\infty} \left(1 + \frac{1}{a} \right) = 1,$$

that is, the improper integral converges.

1543. Find $\int_{-\infty}^{+\infty} \frac{dx}{1 + x^2}$.

Solution. The integrand is even, therefore $\int_{+\infty}^{+\infty} \frac{dx}{1 + x^2} = 2 \int_0^{+\infty} \frac{dx}{1 + x^2}$. Then we have

$$\int_0^{+\infty} \frac{dx}{1 + x^2} = \lim_{b \to +\infty} \int_0^b \frac{dx}{1 + x^2} = \lim_{b \to +\infty} \arctan x \Big|_0^b$$

$$= \lim_{b \to +\infty} \arctan b = \frac{\pi}{2} .$$

Thus, $\int_{-\infty}^{+\infty} \frac{dx}{1 + x^2} = \pi$, that is, the improper integral converges.

1544. Find $\int_0^1 \frac{dx}{x}$.

Solution. The integrand $f(x) = 1/x$ at the point $x = 0$ is unbounded and, therefore, we have

$$\int_0^1 \frac{dx}{x} = \lim_{a \to 0} \int_a^1 \frac{dx}{x} = \lim_{a \to 0} \left| \ln x \right| \Big|_a^1 = \lim_{a \to 0} (\ln 1 - \ln a) = +\infty,$$

that is, the improper integral diverges.

1545. Find $\int\limits_0^{+\infty} xe^{-x^2}dx$.

Solution. We have

$$\int\limits_0^{+\infty} xe^{-x^2}dx = \lim_{b \to +\infty} \int\limits_0^b xe^{-x^2}dx = \lim_{b \to +\infty} \left[-\frac{1}{2} e^{-x^2} \right]_0^b$$

$$= \lim_{b \to +\infty} \left(\frac{1}{2} - \frac{1}{2} e^{-b^2} \right) = \frac{1}{2},$$

that is, the improper integral converges.

Calculate the following improper integrals:

1546. $\int\limits_0^{+\infty} \frac{\arctan x}{1 + x^2} dx$. **1547.** $\int\limits_{-\infty}^{0} \frac{dx}{4 + x^2}$.

1548. $\int\limits_0^2 \frac{x^5 dx}{\sqrt{4 - x^2}}$. **1549.** $\int\limits_0^1 \frac{dx}{\sqrt{x(1 - x)}}$.

1550. $\int\limits_{-1}^1 \frac{dx}{x^2}$. **1551.** $\int\limits_0^1 x \ln^2 x \, dx$. **1552.** $\int\limits_{-\infty}^{+\infty} \frac{dx}{(x^2 + 1)(x^2 + 4)}$.

10.2.2. Comparison tests. To investigate the convergence of improper integrals, use is made of one of the comparison tests.

1. *If the functions $f(x)$ and $\varphi(x)$ are defined for all $x \geqslant a$ and integrable on the closed interval $[a, A]$; where $A \geqslant a$, and if $0 \leqslant f(x) \leqslant \varphi(x)$ for all $x \geqslant a$, then the convergence of the integral* $\int\limits_a^{+\infty} \varphi(x)\,dx$ *yields the convergence of the integral* $\int\limits_a^{+\infty} f(x)\,dx$, *with* $\int\limits_a^{+\infty} f(x)\,dx \leqslant \int\limits_a^{+\infty} \varphi(x)\,dx$.

2. (a) *If for $x \to +\infty$ the function $f(x) \geqslant 0$ is an infinitesimal of the order of smallness $p > 0$ relative to $\frac{1}{x}$, then the integral* $\int\limits_a^{+\infty} f(x)\,dx$ *converges at $p > 1$ and diverges at $p \leqslant 1$.*

(b) *If the function $f(x) \geqslant 0$ is defined and continuous throughout the interval $a \leqslant x < b$ and is an infinitely large quantity of the order p relative to $\frac{1}{b-x}$ as $x \to b - 0$, then the integral $\int_a^b f(x)\,dx$ converges at $p < 1$ and diverges at $p \geqslant 1$.*

1553. Test the integral $\int_a^{+\infty} \frac{dx}{x^p}$ for convergence.

Solution. By definition

$$\int_a^{+\infty} \frac{dx}{x^p} = \lim_{A \to +\infty} \int_a^A x^{-p}dx = \lim_{A \to +\infty} \left[\frac{1}{-p+1} x^{-p+1}\right]_a^A$$

$$= \frac{1}{-p+1} \cdot \lim_{A \to +\infty} A^{-p+1} - \frac{1}{-p+1} \cdot a^{-p+1}.$$

Assume that $p > 1$; then $\lim_{A \to +\infty} A^{-p+1} = 0$. This means that at $p > 1$ the integral converges. Assume now that $p \leqslant 1$; then $\lim_{A \to +\infty} A^{-p+1} = \infty$, that is, the integral $\int_a^{+\infty} \frac{dx}{x^p}$ diverges at $p \leqslant 1$.

1554. Test the integral $\int_0^{+\infty} \sin(x^2)\,dx$ (*Fresnel's integral*) for convergence.

Solution. Assume $x = \sqrt{\tau}$; then $\int_0^{+\infty} \sin(x^2)\,dx = \frac{1}{2} \int_0^{+\infty} \frac{\sin \tau}{\sqrt{\tau}}\,d\tau$. We represent the integral on the right-hand side as a sum:

$$\int_0^{+\infty} \frac{\sin \tau}{\sqrt{\tau}}\,d\tau = \int_0^{\pi/2} \frac{\sin \tau}{\sqrt{\tau}}\,d\tau + \int_{\pi/2}^{+\infty} \frac{\sin \tau}{\sqrt{\tau}}\,d\tau.$$

The first summand is a proper integral since $\lim_{\tau \to 0} \frac{\sin \tau}{\sqrt{\tau}} = 0$, and the second integral must be integrated by parts, with $u = 1/\sqrt{\tau}$, $dv = \sin \tau\,d\tau$:

$$\int_{\pi/2}^{+\infty} \frac{\sin \tau\,d\tau}{\sqrt{\tau}} = -\frac{\cos \tau}{\sqrt{\tau}}\Bigg|_{\pi/2}^{+\infty} - \frac{1}{2} \int_{\pi/2}^{+\infty} \frac{\cos \tau\,d\tau}{\sqrt{\tau^3}} = -\frac{1}{2} \int_{\pi/2}^{+\infty} \frac{\cos \tau\,d\tau}{\tau^{3/2}}.$$

The last integral converges since $\frac{\cos \tau}{\tau^{3/2}} \leqslant \frac{1}{\tau^{3/2}}$, and the integral $\int_{\pi/2}^{+\infty} \frac{d\tau}{\tau^{3/2}}$ con-

verges. Therefore, $\int\limits_0^{+\infty} \frac{\sin \tau}{\sqrt{\tau}}\, d\tau$ converges by virtue of test (2a) and, consequently, the given integral converges as well.

1555. Test the integral $\int\limits_1^{+\infty} \frac{dx}{1 + x^{10}}$ for convergence.

Solution. The integrand $f(x) = 1/(1 + x^{10})$ is smaller over the interval of integration than $\varphi(x) = 1/x^{10}$, and the integral $\int\limits_1^{+\infty} \frac{dx}{x^{10}}$ is convergent. It follows that the given integral converges too.

1556. Test the integral $\int\limits_a^b \frac{dx}{(b - x)^p}$ $(a < b)$ for convergence.

Solution. By definition

$$\int\limits_a^b \frac{dx}{(b - x)^p} = \lim_{\varepsilon \to 0} \int\limits_a^{b-\varepsilon} \frac{dx}{(b - x)^p} = -\frac{1}{-p + 1} \cdot \lim_{\varepsilon \to 0} (b - x)^{-p+1} \Big|_a^{b-\varepsilon}$$

$$= \frac{1}{p - 1} \lim_{\varepsilon \to 0} \varepsilon^{-p+1} + \frac{1}{-p + 1} (b - a)^{-p+1}.$$

If $p < 1$, then $\lim\limits_{\varepsilon \to 0} \varepsilon^{-p+1} = 0$; now if $p > 1$, then $\lim\limits_{\varepsilon \to 0} \varepsilon^{-p+1} = \infty$; if, finally, $p = 1$, then

$$\lim_{\varepsilon \to 0} \int\limits_a^{b-\varepsilon} \frac{dx}{b - x} = -\lim_{\varepsilon \to 0} \ln (b - x) \Big|_a^{b-\varepsilon} = \infty.$$

Consequently, at $p < 1$ the integral $\int\limits_a^b \frac{dx}{(b - x)^p}$ converges and at $p > 1$ it diverges.

1557. Test the integral $\int\limits_0^1 \frac{\cos^2 x}{\sqrt[3]{1 - x^2}}\, dx$ for convergence.

Solution. The integrand is infinitely large when $x \to 1$. Let us represent it in the following form:

$$f(x) = \frac{\cos^2 x}{\sqrt[3]{1 + x}} \cdot \frac{1}{\sqrt[3]{1 - x}} = \frac{\cos^2 x}{\sqrt[3]{1 + x}} \cdot \frac{1}{(1 - x)^{1/3}},$$

that is, as $x \to 1$ the order of this infinitely large function is equal to $p = 1/3 < 1$ relative to $1/(1 - x)$. Therefore, the given integral converges by virtue of test (2b).

1558. Test the integral $\int_0^1 \frac{\ln(1 + \sqrt[3]{x})}{e^{\sin x} - 1} dx$ for convergence.

Solution. The integrand $f(x)$ is positive over the interval of integration and $f(x) \to \infty$ as $x \to 0$. Proceeding from the theorem on equivalent infinitesimals, we transform the numerator and the denominator of the integrand fraction: we have $\ln(1 + \sqrt[3]{x}) \sim x^{1/3}$, and $e^{\sin x} - 1 \sim \sin x$ as $x \to 0$, whence

$$\lim_{x \to 0} \frac{\ln(1 + \sqrt[3]{x})}{e^{\sin x} - 1} = \lim_{x \to 0} \frac{x^{1/3}}{x} = \lim_{x \to 0} \frac{1}{x^{2/3}} = \infty,$$

that is, $f(x)$ is an infinitely large function of the order $p = 2/3$ relative to $1/x$. Consequently, according to test (2b) the given integral converges.

Test the following improper integrals for convergence:

1559. $\int_1^{+\infty} \frac{\ln(1 + x)}{x} dx$. **1560.** $\int_1^{+\infty} \frac{e^{-x^2}}{x^2} dx$.

1561. $\int_0^{+\infty} \frac{x \arctan x}{\sqrt{1 + x^3}} dx$. **1562.** $\int_1^{+\infty} \left(1 - \cos\frac{2}{x}\right) dx$.

1563. $\int_1^{+\infty} \frac{1 + x^2}{x^3} dx$. **1564.** $\int_0^1 \frac{dx}{e^{\sqrt[3]{x}} - 1}$. **1565.** $\int_0^1 \frac{dx}{\tan x - x}$.

10.3. Computing the Area of a Plane Figure

The area of the curvilinear trapezoid bounded by the curve $y = f(x)$ $[f(x) \geqslant 0]$, the straight lines $x = a$ and $x = b$ and by the closed interval [a, b] of the x-axis can be calculated by the formula

$$S = \int_a^b f(x)\, dx.$$

The area of the figure bounded by the curves $y = f_1(x)$ and $y = f_2(x)$ $[f_1(x) \leqslant f_2(x)]$ and by the straight lines $x = a$ and $x = b$ can be found from the formula

$$S = \int_a^b [f_2(x) - f_1(x)]\,dx.$$

If a curve is specified by the parametric equations $x = x(t)$, $y = y(t)$, then the area of the curvilinear trapezoid bounded by this curve, the straight lines $x = a$, $x = b$ and the interval $[a, b]$ of the x-axis is expressed by the formula

$$S = \int_{t_1}^{t_2} y(t)x'(t)\,dt,$$

where t_1 and t_2 can be found from the equations $a = x(t_1)$, $b = x(t_2)$ [$y(t) \geqslant 0$ for $t_1 \leqslant t \leqslant t_2$].

The area of the curvilinear sector bounded by the curve, specified in polar coordinates by the equation $\rho = \rho(\theta)$ and two polar radii $\theta = \alpha$, $\theta = \beta$ ($\alpha < \beta$) is expressed by the integral

$$S = \frac{1}{2}\int_\alpha^\beta \rho^2\,d\theta.$$

1566. Find the area of the figure bounded by the parabola $y = 4x - x^2$ and the x-axis.

Solution. The parabola intersects the Ox axis at the points $O(0; 0)$ and $M(4; 0)$. Consequently,

$$S = \int_0^4 (4x - x^2)\,dx = \left[2x^2 - \frac{1}{3}x^3\right]_0^4 = \frac{32}{2} \text{ (sq. units)}.$$

1567. Calculate the area of the plane figure bounded by one arch of the cycloid $x = 2(t - \sin t)$, $y = 2(1 - \cos t)$ (see Fig. 5) and the Ox axis.

Solution. Here $dx = 2(1 - \cos t)\,dt$, and t varies from $t_1 = 0$ to $t_2 = 2\pi$. Consequently,

$$S = \int_0^{2\pi} 2^2(1 - \cos t)^2 dt = 4\int_0^{2\pi} (1 - 2\cos t + \cos^2 t)dt =$$

$$= 4\left[t - 2\sin t + \frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^{2\pi} = 12 \text{ (sq. units)}.$$

1568. Find the area of the plane figure bounded by the lemniscate $\rho^2 = 2\cos 2\theta$ (see Fig. 2).

Solution. To one-fourth of the desired area there corresponds the variation of θ from 0 to $\pi/4$, and therefore

$$S = 4\cdot\frac{1}{2}\int_0^{\pi/4} 2\cos 2\theta\,d\theta = 2\sin 2\theta\Big|_0^{\pi/4} = 2 \text{ (sq. units)}.$$

Calculate the areas of the figures bounded by the given curves:

1569. $y = -x^2, x + y + 2 = 0.$
1570. $y = 16/x^2, y = 17 - x^2$ (I quadrant).
1571. $y^2 = 4x^3, y = 2x^2.$
1572. $xy = 20, x^2 + y^2 = 41$ (I quadrant).
1573. $y = \sin x, y = \cos x, x = 0.$
1574. $y = 0.25x^2, y = 3x - 0.5x^2.$
1575. $xy = 4\sqrt{2}, x^2 - 6x + y^2 = 0, y = 0, x = 4.$
1576. $x = 12 \cos t + 5 \sin t, y = 5 \cos t - 12 \sin t.$
1577. $x = a \cos^3 t, y = a \sin^3 t.$
1578. $\rho = 4/\cos(\theta - \pi/6), \theta = \pi/6, \theta = \pi/3.$
1579. $\rho = a \cos \theta, \rho = 2a \cos \theta.$
1580. $\rho = \sin^2(\theta/2)$ (on the right of the ray $\theta = \pi/2$).
1581. $\rho = a \sin 3\theta$ (the area of one loop).
1582. $\rho = 2 \cos \theta, \rho = 1$ (outside of the circle $\rho = 1$).

10.4. Computing the Arc Length of a Plane Curve

If the curve $y = f(x)$ is smooth on the interval $[a, b]$ (that is, the derivative $y' = f'(x)$ is continuous), then the length of the corresponding arc of the curve can be found from the formula

$$L = \int_a^b \sqrt{1 + y'^2}\, dx.$$

When the curve $x = x(t)$, $y = y(t)$ is represented parametrically ($x(t)$ and $y(t)$ being continuously differentiable functions) the arc length of the curve corresponding to monotonic variation of the parameter t from t_1 to t_2 can be calculated by the formula

$$L = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt.$$

If a smooth curve is specified in polar coordinates by the equation $\rho = \rho(\theta)$, $\alpha \leqslant \theta \leqslant \beta$, then the arc length is equal to

$$L = \int_\alpha^\beta \sqrt{\rho^2 + \rho'^2}\, d\theta.$$

1583. Find the arc length of the curve $y^2 = x^3$ from $x = 0$ to $x = 1$ ($y \geqslant 0$).

Solution. Differentiating the equation of the curve, we find $y' = (3/2)x^{1/2}$. Thus, we have

$$L = \int_0^1 \sqrt{1 + \frac{9}{4}x}\, dx = \frac{4}{9} \cdot \frac{2}{3} \left(1 + \frac{9}{4}x\right)^{3/2} \Bigg|_0^1$$

$$= \frac{8}{27}\left(\frac{13}{4}\right)^{3/2} - \frac{8}{27} = \frac{8}{27}\left(\frac{13}{8}\sqrt{13} - 1\right).$$

1584. Find the arc length of the curve $x = \cos^5 t$, $y = \sin^5 t$ from $t_1 = 0$ to $t_2 = \pi/2$.

Solution. Let us find the derivatives with respect to the parameter t: $\dot{x} = -5\cos^4 t \sin t$, $\dot{y} = 5\sin^4 t \cos t$. Consequently,

$$L = \int_0^{\pi/2} \sqrt{(-5\cos^4 t \sin t)^2 + (5\sin^4 t \cos t)^2}\, dt = 5\int_0^{\pi/2} \sin t \cos t \sqrt{\sin^6 t + \cos^6 t}\, dt$$

$$= \frac{5}{2}\int_0^{\pi/2} \sin 2t \sqrt{\frac{1}{4} + \frac{3}{4}\cos^2 2t}\, dt = -\frac{5}{8}\int_0^{\pi/2} \sqrt{1 + 3\cos^2 2t}\, d(\cos 2t)$$

$$= -\frac{5}{8\sqrt{3}}\left[\frac{\sqrt{3}}{2}\cos 2t\sqrt{1 + 3\cos^2 2t} + \frac{1}{2}\ln(\sqrt{3}\cdot\cos 2t\right.$$

$$\left.+ \sqrt{1 + 3\cos^2 2t})\right]_0^{\pi/2} = \frac{5}{8}\left[2 - \frac{\ln(2 - \sqrt{3})}{\sqrt{3}}\right].$$

1585. Find the arc length of the curve $\rho = \sin^3(\theta/3)$ from $\theta_1 = 0$ to $\theta_2 = \pi/2$.

Solution. We have $\rho' = \sin^2(\theta/3)\cos(\theta/3)$. Consequently,

$$L = \int_0^{\pi/2} \sqrt{\sin^6\frac{\theta}{3} + \left(\sin^2\frac{\theta}{3}\cos\frac{\theta}{3}\right)^2}\, d\theta = \int_0^{\pi/2} \sin^2\frac{\theta}{3}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(1 - \cos\frac{2\theta}{3}\right) d\theta = \frac{1}{2}\left[\theta - \frac{3}{2}\sin\frac{2\theta}{3}\right]_0^{\pi/2} = \frac{1}{8}(2\pi - 3\sqrt{3}).$$

Calculate the arc lengths of the following curves:

1586. $y = \ln \sin x$ from $x = \pi/3$ to $x = \pi/2$.

1587. $y = (2/5)x^4\sqrt{x} - (2/3)^4\sqrt{x^3}$ between the points of intersection with the Ox axis.

1588. $y = x^2/2$ from $x = 0$ to $x = 1$.

1589. $y = 1 - \ln \cos x$ from $x = 0$ to $x = \pi/6$.

1590. $y = \cosh x$ from $x = 0$ to $x = 1$.

1591. $x = t^3/3 - t$, $y = t^2 + 2$ from $t = 0$ to $t = 3$.

1592. $x = e^t \cos t$, $y = e^t \sin t$ from $t = 0$ to $t = \ln \pi$.

1593. $x = 8\sin t + 6\cos t$, $y = 6\sin t - 8\cos t$ from $t = 0$ to $t = \pi/2$.

1594. $x = 9(t - \sin t)$, $y = 9(1 - \cos t)$ (the arc length of one arch of the cycloid).

1595. $\rho = \theta^2$ from $\theta = 0$ to $\theta = \pi$.

1596. $\rho = a \sin \theta$.

1597. $\rho = a\cos^3(\theta/3)$ from $\theta = 0$ to $\theta = \pi/2$.
1598. $\rho = 1 - \cos\theta$.

10.5. Computing the Volume of a Body

10.5.1. Computing the volume of a body from the known areas of cross sections. If the area of the section of a body by a plane perpendicular to the x-axis can be expressed as a function of x, that is, in the form $S = S(x)$, $a \leqslant x \leqslant b$, then the volume of the part of the body, contained between the planes $x = a$ and $x = b$ which are perpendicular to the x-axis, can be found from the formula

$$V = \int_a^b S(x)\,dx.$$

10.5.2. Computing the volume of a body of revolution. If a curvilinear trapezoid, bounded by the curve $y = f(x)$ and the straight lines $y = 0$, $x = a$, $x = b$, is rotating about the Ox axis, the volume of the body of revolution can be found from the formula

$$V_x = \pi\int_a^b y^2\,dx.$$

If the figure bounded by the curves $y_1 = f_1(x)$ and $y_2 = f_2(x)$ $[0 \leqslant f_1(x) \leqslant f_2(x)]$ and the straight lines $x = a$, $x = b$ is rotating about the Ox axis, then the volume of the body of revolution is

$$V_x = \pi\int_a^b (y_2^2 - y_1^2)\,dx.$$

1599. Find the volume of the body generated by rotation about the x-axis of a figure bounded by the curve $y^2 = (x - 1)^3$ and the straight line $x = 2$ (Fig. 43).

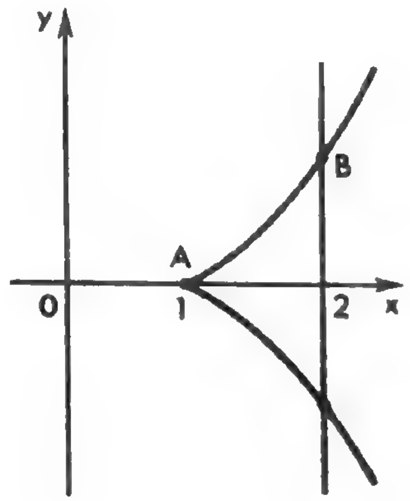

Fig. 43

Solution. We have

$$V = \pi \int_1^2 y^2 dx = \pi \int_1^2 (x-1)^3 dx = \frac{1}{4}\pi (x-1)^4 \Big|_1^2 = \frac{1}{4}\pi \text{ (cubic units)}.$$

1600. Find the volume of the body resting on an isosceles triangle with the altitude h and the base a. The cross section of the body is a segment of a parabola with the chord equal to the altitude of the segment (Fig. 44).

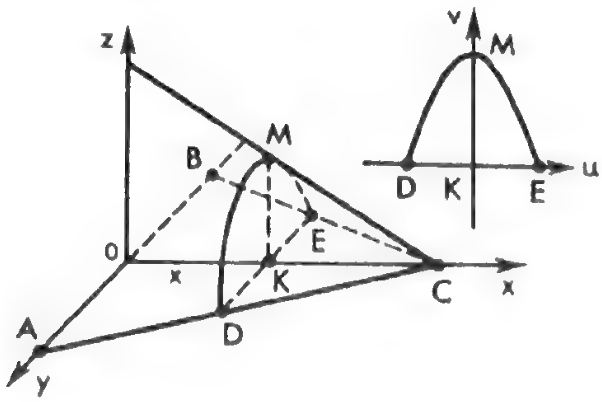

Fig. 44

Solution. We have $|AB| = a$, $|OC| = h$, $|MK| = |DE|$, $|OK| = x$. Let us express the area of the cross section as a function of x, for which purpose we shall first find the equation of the parabola. The length of the chord DE can be found from the similarity of the corresponding triangles, namely,

$$|DE|/a = (h - x)/h, \quad \text{i.e.} \quad |DE| = a(h - x)/h = |MK|.$$

If we set $|DE| = m$, the equation of the parabola, in the coordinate system uKv, assumes the form $v = m - \dfrac{4}{m} u^2$. From this we find the area of the cross section of the given body:

$$S = 2 \int_0^{m/2} \left(m - \frac{4}{m} u^2 \right) du = \frac{2}{3} m^2, \quad \text{or} \quad S(x) = \frac{2}{3} \cdot \frac{a^2 (h - x)^2}{h^2}.$$

Thus, we have

$$V = \int_0^h S(x)\, dx = \int_0^h \frac{2}{3} \cdot \frac{a^2}{h^2} (h - x)^2\, dx = \frac{2}{9} a^2 h.$$

Find the volumes of bodies generated by rotation about the Ox axis of figures bounded by the following curves:

1601. $y = \dfrac{64}{x^2 + 16}$, $x^2 = 8y$.

1602. $y^2 = x$, $x^2 = y$.

1603. $y = \sqrt{x}\, e^x$, $x = 1$, $y = 0$.

1604. $y = x^2/2$, $y = x^3/8$.

1605. Find the volume of the body bounded by the planes $x = 1$, $x = 3$ if the area of its cross section is inversely proportional to the square of the distance of the section from the origin, and at $x = 2$ the area of the section is equal to 27 (sq. units).

1606. Find the volume of the cylindrical wedge from its dimensions given in Fig. 45 (Archimedean problem).

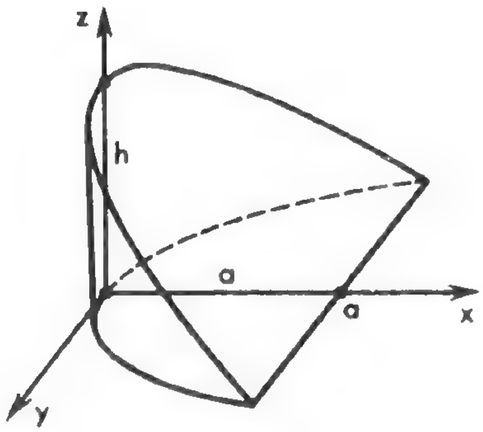

Fig. 45

1607. A paraboloid of revolution is inserted into a cylindrical cup the vertex down. The base and the altitude of the paraboloid coincide with those of the cylinder. Find the volume of the water remaining in the cup if the radius of the base is r and the altitude is h.

10.6. Computing the Area of a Surface of Revolution

If the arc of the smooth curve $y = f(x)$ $(a \leqslant x \leqslant b)$ is rotating about the x-axis, then the area of the surface of revolution can be calculated by the formula

$$S_x = 2\pi\int_a^b y\sqrt{1 + y'^2}\,dx.$$

If the curve is specified by the parametric equations $x = x(t)$, $y = y(t)$ $(t_1 \leqslant t \leqslant t_2)$, then

$$S_x = 2\pi \int_{t_1}^{t_2} y\sqrt{\dot{x}^2 + \dot{y}^2}\,dt.$$

1608. Find the area of the surface generated by rotation about the x-axis of the arc of the sine curve $y = \sin 2x$ from $x = 0$ to $x = \pi/2$.

Solution. We find $y' = 2\cos 2x$; then we have

$$S_x = 2\pi \int_0^{\pi/2} \sin 2x\sqrt{1 + 4\cos^2 2x}\,dx.$$

We perform a change of variable: $2 \cos 2x = t$, $-4 \sin 2x\, dx = dt$, $\sin 2x\, dx = (-1/4)dt$.

Next we find the limits of integration with respect to t: if $x = 0$, then $t = 2$; if $x = \pi/2$, then $t = -2$.

Thus we have

$$S = 2\pi \int_{2}^{-2} \sqrt{1+t^2}\left(-\frac{1}{4}\right) dt = \frac{\pi}{2} \int_{-2}^{2} \sqrt{1+t^2}\, dt$$

$$= \frac{\pi}{2}\left[\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\ln\left(t + \sqrt{1+t^2}\right)\right]_{-2}^{2}$$

$$= \frac{\pi}{2}\left(2\sqrt{5} + \frac{1}{2}\ln\frac{\sqrt{5}+2}{\sqrt{5}-2}\right) = \frac{\pi}{2}\left[2\sqrt{5} + \ln(\sqrt{5}+2)\right] \text{ (sq. units).}$$

Find the areas of the surfaces generated by rotation of the arcs of the following curves about the Ox axis:

1609. $y = 2\cosh(x/2)$ from $x = 0$ to $x = 2$.
1610. $y = x^3$ from $x = 0$ to $x = 1/2$.
1611. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.
1612. $x = t - \sin t$, $y = 1 - \cos t$ (the area generated by rotation of one arch).

10.7. Static Moments and Moments of Inertia of Plane Arcs and Figures

Suppose a system of particles $A_1(x_1; y_1)$, $A_2(x_2; y_2)$, . . ., $A_n(x_n; y_n)$ with masses $m_1, m_2, \ldots, m_n$ are given on the plane xOy. The *static moment* M_x of the system about the x-axis is the sum of the products of the masses of these particles by their ordinates:

$$M_x = \sum_{k=1}^{k=n} m_k y_k.$$

The static moment of the system about the y-axis can be determined by a complete analogy (as the sum of the products of the masses of the particles by their abscissas):

$$M_y = \sum_{k=0}^{k=n} m_k x_k.$$

The *moments of inertia* I_x and I_y of the system about the x- and y-axes are the sums of the products of the masses of the particles by the squares of their distances

from the respective axis. Thus, we have

$$I_x = \sum_{k=1}^{k=n} m_k y_k^2; \quad I_y = \sum_{k=1}^{k=n} m_k x_k^2.$$

The static moments and the moments of inertia of plane arcs and figures are assumed to be the corresponding moments of conventional masses distributed uniformly along these arcs and figures with unit density (linear or areal).

The static moments and the moments of inertia of an arc of the plane curve $y = f(x)$ $(a \leqslant x \leqslant b)$ can be calculated by the formulas

$$M_x = \int_a^b y\,dL; M_y = \int_a^b x\,dL; \quad I_x = \int_a^b y^2\,dL; \quad I_y = \int_a^b x^2\,dL,$$

where $dL = \sqrt{1 + y'^2}\,dx$ is the differential of the arc of the curve.

The static moments and the moments of inertia of the curvilinear trapezoid bounded by the curve $y = f(x)$, the x-axis and the two straight lines $x = a$ and $x = b$, can be calculated by the formulas

$$M_x = \frac{1}{2}\int_a^b y\,dS = \frac{1}{2}\int_a^b y^2\,dx, \quad M_y = \int_a^b x\,dS = \int_a^b xy\,dx,$$

$$I_x = \frac{1}{3}\int_a^b y^3\,dx, \quad I_y = \int_a^b x^2\,dS = \int_a^b x^2y\,dx.$$

In these formulas $dS = y\,dx$ is the differential of the area of the curvilinear trapezoid.

1613. Find the static moment and the moment of inertia of the semicircle $y = \sqrt{r^2 - x^2}$ $(-r \leqslant x \leqslant r)$ about the Ox axis.

Solution. We shall compute the static moment M_x by the formula $M_x = \int_a^b y\,dL$, where $dL = \sqrt{1 + y'^2}\,dx$, $y' = -x/\sqrt{r^2 - x^2}$. Then we shall obtain

$$M_x = \int_{-r}^{r} \sqrt{r^2 - x^2}\sqrt{1 + \frac{x^2}{r^2 - x^2}}\,dx = r\int_{-r}^{r} dx = 2r^2.$$

Next we find the moment of inertia about the x-axis:

$$I_x = \int_a^b y^2\,dL = \int_{-r}^{r} (r^2 - x^2)\sqrt{1 + \frac{r^2}{r^2 - x^2}}\,dx$$

$$= r\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx = 2r\int_0^r \sqrt{r^2 - x^2}\,dx.$$

Then we shall make the substitution $x = r \sin t$, $dx = r \cos t\,dt$; if $x = 0$, then $t = 0$; now if $x = r$, then $t = \pi/2$. Consequently,

$$I_x = 2r\int_0^{\pi/2} \sqrt{r^2 - r^2\sin^2 t}\; r\cos t\,dt$$

$$= r^3\int_0^{\pi/2} (1 + \cos 2t)\,dt = r^3\left[t + \frac{1}{2}\sin 2t\right]_0^{\pi/2} = \frac{\pi r^3}{2}.$$

1614. Find the moment of inertia of the area of the ellipse $x = a\cos t, y = b\sin t$ about the y-axis.

Solution. The moment of inertia of the area of the ellipse about the y-axis is equal to $I_y = \int_{-a}^{a} x^2\,dS$, where $dS = 2y\,dx$. We find from the parametric equations of the ellipse that $dS = 2b\sin t \cdot a(-\sin t)\,dt = -2ab\sin^2 t\,dt$, whence

$$I_y = 2\int_{\pi/2}^{0} a^2\cos^2 t(-2ab\sin^2 t)\,dt = -4a^3b\int_{\pi/2}^{0}\sin^2 t\cos^2 t\,dt$$

$$= \frac{1}{2}a^3b\int_{0}^{\pi/2}(1-\cos 4t)\,dt = \frac{\pi a^3 b}{4}.$$

1615. Find the static moments and the moments of inertia of an arc of the astroid $x = a\cos^3 t, y = a\sin^3 t$, lying in the first quadrant (Fig. 46).

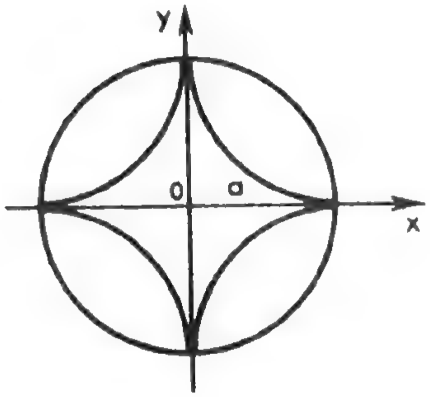

Fig. 46

Solution. By virtue of the symmetry of the astroid about the coordinate axes we have $M_x = M_y$, $I_x = I_y$. Therefore, it is sufficient to calculate the moments about the x-axis. For the first quadrant we have $0 \leqslant t \leqslant \pi/2$. We find

$$dL = \sqrt{x_t'^2 + y_t'^2}\,dt = 3a\sin t\cos t\,dt,$$

$$M_x = \int_a^b y\,dL = \int_0^{\pi/2} a\cdot\sin^3 t\cdot 3a\sin t\cos t\,dt = \frac{3a^2}{5}\sin^5 t\Big|_0^{\pi/2} = \frac{3}{5}a^2,$$

$$I_x = \int_a^b y^2\,dL = \int_0^{\pi/2} a^2\sin^6 t\cdot 3a\sin t\cos t\,dt = \frac{3}{8}a^3\sin^8\Big|_0^{\pi/2} = \frac{3}{8}a^3.$$

Thus, we have $M_x = M_y = (3/5)a^2$; $I_x = I_y = (3/8)a^3$.

1616. Find the moment of inertia of a parabolic segment whose chord is a and the arrow relative to the chord is h (Fig. 47).

Solution. We have $|AB| = a$, $|OC| = h$. The equation of the parabola is written in the form $y = h - Nx^2$, where the undetermined coefficient N can be found

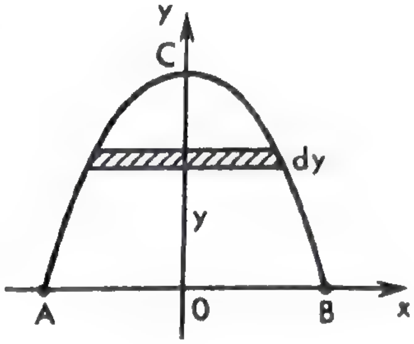

Fig. 47

using the fact that the point $B(a/2; 0)$ belongs to the parabola: $0 = h - Na^2/4$, or $N = 4h/a^2$; hence, $y = h - 4hx^2/a^2$. Now we find the desired moment of inertia:

$$I_x = \frac{1}{3}\int_{-a/2}^{a/2} y^3\,dx = \frac{2}{3}\int_0^{a/2}\left(h - \frac{4h}{a^2}x^2\right)^3 dx = \frac{16}{105}ah^3.$$

1617. Find the static moment and the moment of inertia of an arc of the catenary $y = \frac{a}{2}(e^{x/a} + e^{-x/a})$, where $0 \leqslant x \leqslant a$, about the x-axis.

1618. Find the static moment and the moment of inertia of a triangle with base a and altitude h about its base.

1619. Find the moment of inertia of the parabolic segment bounded by the parabola $y = 4 - x^2$ and the straight line $y = 3$ about the x-axis.

1620. Find the moment of inertia of a rectangle with sides a and b about the axes of symmetry of the rectangle.

1621. Find the polar moment of inertia of a circle of diameter d, that is, the moment of inertia about the axis passing through the centre of the circle at right angles to its plane.

10.8. Finding the Coordinates of the Centre of Gravity. Guldin's Rule

The coordinates of the centre of gravity of a homogeneous arc of the plane curve $y = f(x)$ $(a \leqslant x \leqslant b)$ are expressed by the formulas

$$\bar{x} = \frac{1}{L}\int_a^b x\,dL, \quad \bar{y} = \frac{1}{L}\int_a^b y\,dL,$$

where $dL = \sqrt{1 + y'^2}\,dx$, and L is the arc length.

The coordinates of the centre of gravity of a curvilinear trapezoid can be found from the formulas

$$\bar{x} = \frac{1}{S}\int_a^b x\,dS = \frac{1}{S}\int_a^b xy\,dx, \quad \bar{y} = \frac{1}{2S}\int_a^b y\,dS = \frac{1}{2S}\int_a^b y^2\,dx,$$

where $dS = y\,dx$, and S is the area of the figure.

Guldin's Rules

Rule 1. *The area of the surface obtained as a result of rotating an arc of a plane curve about an axis lying in the plane of that curve and not cutting it is equal to the length of the arc of the curve multiplied by the length of a circle described by the centre of gravity of the arc.*

Rule 2. *The volume of the body obtained as a result of rotating a plane figure about an axis lying in the plane of the figure and not cutting it is equal to the poduct of the area of that figure by the length of the circle described by the centre of gravity of the figure.*

1622. Find the coordinates of the centre of gravity of the catenary $y = a \cosh (x/a)$, $-a \leqslant x \leqslant a$.

Solution. The curve being symmetric about the y-axis, its centre of gravity lies on that axis, i.e. $\bar{x} = 0$. It remains to find $\bar{y}$. We have $y' = \sinh (x/a)$; then $dL = \sqrt{1 + \sinh^2 (x/a)}\, dx = \cosh (x/a)\, dx$; the arc length

$$L = \int_{-a}^{a} \sqrt{1 + y'^2}\, dx = 2 \int_0^a \cosh \frac{x}{a}\, dx = 2a \sinh \frac{x}{a} \Big|_0^b = 2a \sinh 1.$$

Consequently,

$$\bar{y} = \frac{1}{2a \sinh 1} \int_{-a}^{a} a \cosh^2 \frac{x}{a}\, dx = \frac{1}{\sinh 1} \int_0^a \cosh^2 \frac{x}{a}\, dx = \frac{1}{2 \sinh 1} \int_0^a \left(1 + \cosh \frac{2x}{a}\right) dx$$

$$= \frac{1}{2 \sinh 1} \left[x + \frac{a}{2} \sinh \frac{2x}{a}\right]_0^a = \frac{a}{2 \sinh 1} \left(1 + \frac{1}{2} \sinh 2\right)$$

$$= \frac{a(2 + \sinh 2)}{4 \sinh 1} \approx 1.18a.$$

1623. Find the coordinates of the centre of gravity of the figure bounded by an arc of the ellipse $x = a \cos t$, $y = b \sin t$, lying in the first quadrant, and by the coordinate axes.

Solution. In the first quadrant, the quantity t decreases from $\pi/2$ to 0 as x increases from 0 to a; therefore,

$$\bar{x} = \frac{1}{S} \int_0^a xy\, dx = \frac{1}{S} \int_{\pi/2}^{0} a \cos t \cdot b \sin t\, (-a \sin t)\, dt$$

$$= \frac{a^2 b}{S} \int_0^{\pi/2} \sin^2 t \cos t\, dt = \frac{a^2 b}{S}\, \frac{1}{3} \sin^3 t \Big|_0^{\pi/2} = \frac{a^2 b}{3S}.$$

Applying the formula for the area of an ellipse $S = \pi ab$, we get $\bar{x} = (4a^2 b)(3\pi ab) = (4a)/(3\pi)$.

By analogy we find

$$\bar{y} = \frac{1}{2S}\int_0^a y^2\,dx = \frac{1}{2S}\int_{\pi/2}^{0} b^2 \sin^2 t\,(-a \sin t)\,dt$$

$$= \frac{2ab^2}{\pi ab}\int_{\pi/2}^{0} (1 - \cos^2 t)\,d(\cos t) = \frac{2b}{\pi}\left[\cos t - \frac{1}{3}\cos^3 t\right]_{\pi/2}^{0} = \frac{4b}{3\pi}.$$

Thus, we have $\bar{x} = 4a/(3\pi)$, $\bar{y} = 4b/(3\pi)$.

1624. Find the areas of the surfaces and the volumes of the anchor rings (tori) generated by rotating the circle $(x - a)^2 + (y - b)^2 \leqslant r^2$ about the x- and y-axes $(a \geqslant r, b \geqslant r)$.

Solution. If the circle rotates about the x-axis, then the centre of gravity of the circle is at the distance b from the axis of revolution. Therefore, in accordance with the first Guldin Rule, the area of the surface is equal to $S_x = 2\pi r \cdot 2\pi b = 4\pi^2 br$, and in accordance with the second Guldin Rule, the volume is equal to $V_x = \pi r^2 \cdot 2\pi b = 2\pi^2 br^2$.

Now if the circle rotates about the y-axis, then the distance between the centre of gravity of the circle and the y-axis is a. Then, $S_y = 2\pi r \cdot 2\pi a = 4\pi^2 ar$, $V_y = \pi r^2 \cdot 2\pi a = 2\pi^2 ar^2$.

1625. Making use of Guldin's theorem, find the coordinates of the centre of gravity of the quarter-circle $x^2 + y^2 \leqslant r^2$.

Solution. Rotating the quarter-circle about the x-axis, we get a semisphere whose volume is equal to $V = (1/2) \cdot (4\pi r^3/3) = 2\pi r^3/3$. In accordance with Guldin's second theorem, $V = (\pi r^2/4) \cdot (2\pi \bar{y})$, whence, $\bar{y} = 2V/(\pi^2 r^2) = 2 \times 2\pi r^3/(3\pi^2 r^2) = 4r/(3\pi)$. The centre of gravity of the quarter-circle lies on the symmetry axis, that is, on the bisector of the 1st quadrant, and therefore $\bar{x} = \bar{y} = 4r/(3\pi)$.

1626. Find the coordinates of the centres of gravity of the semicircle $y = \sqrt{r^2 - x^2}$ and half a disc bounded by this semicircle and the x-axis.

1627. Find the coordinates of the centre of gravity of the figure bounded by the lines $x = 0$, $x = \pi/2$, $y = 0$, $y = \cos x$.

1628. Find the coordinates of the centre of gravity of the parabolic segment bounded by the lines $y = 4 - x^2$, $y = 0$.

1629. Find the coordinates of the centre of gravity of an arc of the astroid $x = a \cos^3 t$, $y = a \sin^3 t$ (in the first quadrant).

1630. Find the coordinates of the centre of gravity of the figure bounded by the lines $y = 2x - x^2$, $y = 0$.

1631. Find the coordinates of the centre of gravity of the figure bounded by the lines $x = 0$, $x = \pi/2$, $y = 0$, $y = \sin x$.

1632. Using Guldin's Rule, find the volume of the body generated by rotating a semircircle of radius r about a tangent line which is parallel to the diameter.

1633. Using Guldin's Rule prove that the centre of gravity of a triangle is at a distance equal to one-third its altitude from its base.

Hint. Find the volume of the body obtained as a result of rotating the triangle about the base.

1634. Using Guldin's Rule, find the volume of the body generated by rotating the rectangle with sides 6 and 8 about the axis passing through its vertex at right angles to the diagonal.

10.9. Computation of Work and Pressure

The *work done by the variable force* $X = f(x)$ directed along the Ox axis on the closed interval $[x_0, x_1]$ can be computed by the formula

$$A = \int_{x_0}^{x_1} f(x)\,dx.$$

To compute the *pressure of liquid*, use can be made of Pascal's law which allows a statement to be made that the pressure of liquid on a plane surface is equal to its area S multiplied by the depth of immersion h, by liquid density ρ and by the acceleration of gravity g, i.e.

$$P = \rho g h S.$$

1635. What work must be done in order to extend the spring by 4 cm, if it is known that a load of 1 N produces an extension of 1 cm?

Solution. In accordance with Hooke's law, the force of X N extending the spring by x m is equal to $X = kx$. The proportionality factor k can be found from the following condition: if $x = 0.01$ m, then $X = 1$ N; it follows that $k = 1/0.01 = 100$ and $X = 100x$. Then

$$A = \int_{0}^{0.04} 100x\,dx = 50x^2 \Big|_{0}^{0.04} = 0.08 \text{ J}.$$

1636. What work is done by a crane extracting a reinforced concrete block from the bottom of the river 5 m deep if the block is shaped as a regular tetrahedron with an edge of 1 m, and the density of reinforced concrete is 2500 kg/m^3 (Fig. 48)?

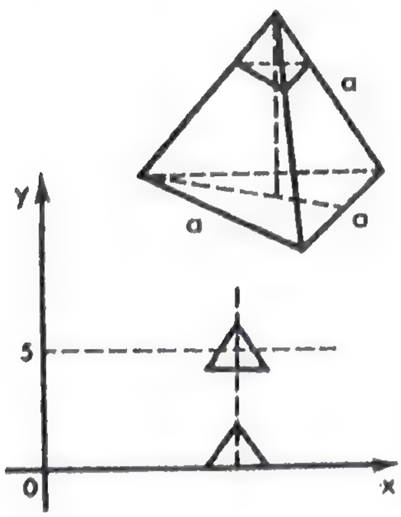

Fig. 48

Solution. The altitude of the tetrahedron $h = \sqrt{6}/3$ m, its volume $V = \sqrt{2}/12$ m^3. The weight of the block in water

$$P = (1/12) \cdot \sqrt{2} \cdot 2500 \cdot 9.8 - (1/12) \cdot \sqrt{2} \cdot 1000 \cdot 9.8 = 1225\sqrt{2} \text{ N},$$

therefore, the work done to raise the block until its vertex shows on the water surface amounts to

$$A_0 = 1225\sqrt{2}\,(5 - h) = 1225\sqrt{2}\,(5 - \sqrt{6}/3) \cong 7227 \cdot 5 \text{ J}.$$

Now we shall compute the work A_1 done to extract the block from the water. Suppose the vertex of the tetrahedron has been raised to a height of $5 + y$, then the volume of the small tetrahedron that appeared from the water is $3\sqrt{3}y^3/8$, and the weight of the tetrahedron

$$P(y) = \frac{2500 \cdot 9.8}{12}\sqrt{2} - \left(\frac{1}{12}\sqrt{2} - \frac{1}{8}y^3 3\sqrt{3}\right) 1000 \cdot 9.8 \text{ N}.$$

Consequently,

$$A_1 = \int_0^h \left(\frac{24\,500}{12}\sqrt{2} - \frac{9800}{12}\sqrt{2} + \frac{9800}{8}3y^3\sqrt{8}\right) dy$$

$$= \int_0^{\sqrt{6}/3} (1225\sqrt{2} + 3675\sqrt{3}y^3)\, dy = \left[1225\sqrt{2}y + \frac{3675}{4}\sqrt{3}y^4\right]_0^{\sqrt{6}/3} \approx 2082.5 \text{ J}.$$

Hence,

$$A = A_0 + A_1 = 7227.5 + 2082.5 = 9310 \text{ J} = 9.31 \text{ kJ}.$$

1637. Calculate the work done to pump water out of a trough shaped as a half-cylinder whose length is a and radius is r (Fig. 49).

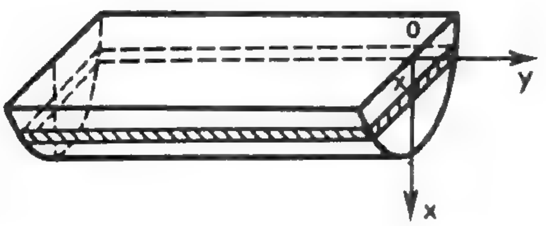

Fig. 49

Solution. The volume of the elementary water layer at the depth x, a in length, $m = 2\sqrt{r^2 - x^2}$ wide and dx thick, is equal to

$$dV = am\, dx = 2a\sqrt{r^2 - x^2}\, dx.$$

An element of work done to raise this water layer to the height x is $dA = 2\rho gax\sqrt{r^2 - x^2}\, dx$, where ρ is the density of water. Consequently,

$$A = 2a\rho g \int_0^r x\sqrt{r^2 - x^2}\, dx = -a\rho g\left[\frac{2}{3}(r^2 - x^2)^{3/2}\right]_0^r = \frac{2}{3}\rho gar^3.$$

1638. A water-supply pipe has a diameter of 6 cm; one of its ends is connected with a tank, the water level in which is 100 cm higher than the upper edge of the pipe, and the other end is closed by means of a gate valve. Find the total pressure exerted on the gate valve.

Solution. The gate valve is shaped as a circle whose radius is 3 cm. Let us divide the area of the circle into elements, i.e. strips parallel to the water surface. The area of one such element which is at the distance y from the centre is equal (with an accuracy to within infinitesimals of the higher order of smallness) to $dS = 2\sqrt{9 - y^2}\,dy$ cm^2. Let us calculate the pressure exerted on the element:

$$dP = 2\rho g\,(103 - y)\sqrt{9 - y^2}\,dy = 196(103 - y)\sqrt{9 - y^2}\,dy \text{ N/m}^2$$

(here $\rho = 1$ g/cm^3). Consequently,

$$P = 196\int_{-3}^{3}(103 - y)\sqrt{9 - y^2}\,dy$$

$$= 196\left[103\left(\frac{y}{2}\sqrt{9 - y^2} + \frac{9}{2}\arcsin\frac{y}{3}\right) + \frac{1}{3}(9 - y^2)^{3/2}\right]_{-3}^{3} = 90.846\pi \text{ kPa}.$$

1639. Find the pressure of water on a vertical wall shaped as a semicircle whose diameter, equal to 6m, is on the surface of the water (Fig. 50). Water density $\rho = 1000$ kg/m^3.

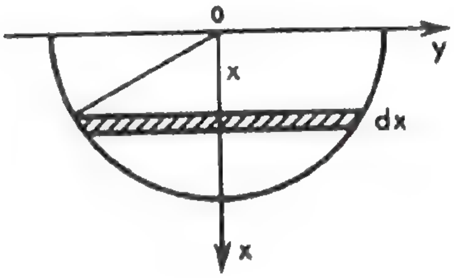

Fig. 50

Solution. The differential of pressure on the element of the area is expressed as follows:

$$dP = 2\rho\,gx\sqrt{9 - x^2}\,dx = 19600x\sqrt{9 - x^2}\,dx.$$

Hence,

$$P = 19600\int_{3}^{3} x\sqrt{9 - x^2}\,dx = -\frac{19600}{3}(9 - x^2)^{3/2}\Big|_0^3 = 176.4 \text{ kPa}.$$

1640. Find the pressure of petrol contained in a cylindrical tank with the altitude $h = 3.5$ m and the base radius $r = 1.5$ m on the walls of the tank if $\rho = 900$ kg/m^3.

Solution. The element of pressure on the surface of the wall in the isolated strip

can be expressed as $dP = \rho g \cdot 2\pi r x\, dx$. Hence,

$$P = 2\pi r \rho g \int_0^h x\, dx = \rho g \pi r h^2 = 9.8\pi \cdot 1.5 \cdot 3.5^2 \cdot 900 = 162.07\pi \text{ kPa}.$$

1641. What pressure is exerted on a rectangular plate a in length and b wide ($a > b$), if it is at an angle α to the horizontal of the liquid and its larger side is at a depth h (Fig. 51)?

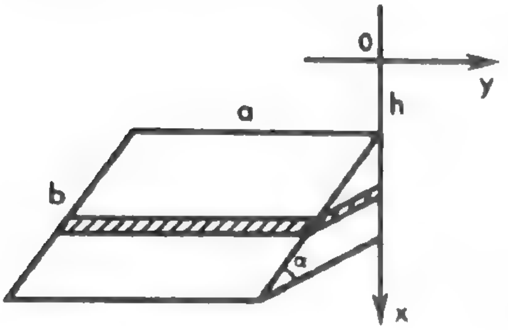

Fig. 51

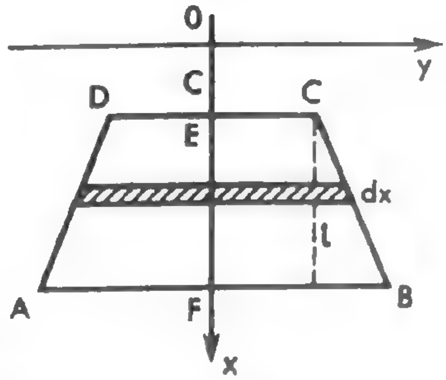

Fig. 52

Solution. The area of the elementary strip isolated at a depth x is $dS = (a/\sin\alpha)\, dx$. Consequently, the element of pressure $dP = (ax\rho g/\sin\alpha)\, dx$ (ρ is the density of the liquid). From this we find

$$P = a\rho g \int_h^{h+b\sin\alpha} \frac{x dx}{\sin\alpha} = \frac{a\rho g}{\sin\alpha} \cdot \frac{1}{2} x^2 \Big|_h^{h+b\sin\alpha}$$

$$= \frac{a\rho g}{2\sin a}[(h^2 + 2bh\sin\alpha + b^2\sin^2\alpha) + h^2] = ab\rho g\left(h + \frac{1}{2}b\sin\alpha\right).$$

1642. Find the pressure exerted on a plate shaped as an equilateral trapezoid with the bases a and b and the altitude h, immersed into a liquid to the depth c (Fig. 52).

Solution. The area of the elementary strip is expressed as follows: $dS = (a + 2l)dx$, where $l = (b - a)(x - c)/(2h)$ (l is determined from the similarity of the triangles). Consequently, we have

$$P = \rho g \int_c^{c+h} \left[a + \frac{b-a}{h}(x - c)\right] x\, dx$$

$$= \rho g \left[\frac{ax^2}{2} + \frac{b-a}{h}\left(\frac{1}{3}x^3 - \frac{cx^2}{2}\right)\right]_c^{c+h} = \left[\frac{a+b}{2} ch + \frac{h^2}{6}(a + 2b)\right] \rho g.$$

1643. Find the work done to pump water out of a conical vessel resting on its horizontal base if the radius of the base is r and the altitude of the vessel is h.

1644. Liquid is being pumped out of a cylindrical tank. What amount of work should be done if the length of the tank is a and the diameter is d?

1645. A conical stone is being raised from the water by means of a crane and a

rope attached to the vertex of the stone. Calculate the work which should be done to complete the operation if originally the vertex of the cone was at the water surface. The radius of the cone base is 1 m, its altitude is 3 m, its density is 2.5 g/cm^3.

Hint. Here $P(y) = 14700\pi + (9800/27)\pi y^3$ N.

1646. A cast-iron right circular cone 40 cm in height and with the radius of the base equal to 40 cm is on the bottom of a basin filled up with petrolium oil to the brim. Calculate the work that should be done to take the cone from the basin if the density of the cast iron $\rho_1 = 7.22$ g/cm^3 and the density of the petroleum oil $\rho_2 = 0.89$ g/cm^3.

Hint. Here $P(y)$ is equal to the weight of the cone without the weight of the mass of oil displaced by the submerged portion of the cone, i.e. $P(y) = \left[\frac{1}{3}\pi\cdot 40^3\rho_1 g - \left(\frac{1}{3}\pi\cdot 40^3 - \frac{1}{3}\pi y^3\right)\rho_2 g\right] 10^{-5}$ N.

1647. A cylinder 24 cm in diameter and 80 cm in length is filled up with gas under the pressure of 2 kPa. What work must be done to compress the gas isothermally to the volume half as large?

1648. A triangular plate is immersed the vertex upwards into a liquid with density ρ. Find the pressure of the liquid on the plate if the base of the triangle is a and the altitude is h. The vertex of the triangle is at the surface.

1649. Find the pressure of the petroleum contained in a cylindrical tank with the altitude $h = 4$ m and the radius $r = 2$ m ($\rho = 900$ kg/m^3) on the walls of the tank at each metre of the depth.

1650. A circular plate d in diameter is immersed into a liquid with density ρ, the plate touching the surface of the liquid. Find the pressure of the liquid on the plate.

Setting up the requisite integral sums and passing to the limit, solve the following problems:

1651. Find the mass of a bar 100 cm long if the linear density of the bar varies in accordance with the law $\delta = (20x + 0.15x^2)$ g/cm, where x is the distance from one of the ends of the bar.

1652. The velocity of the point varies according to the law $v = (100 + 8t)$ m/s. What path will the point traverse in the time interval $[0, 10]$?

1653. A point moves along the Ox axis, starting from the point $M(1; 0)$, so that its velocity is its abscissa. Where will it be 10 seconds from the beginning of the motion?

1654. The velocity of the point varies according to the law $v = 2(6 - t)$ m/s. What is the greatest distance of the moving point from the initial point?

10.10. Some Data on Hyperbolic Functions

Hyperbolic functions are functions specified by the equations

$$\sinh x = \frac{e^x - e^{-x}}{2}, \text{ hyperbolic sine,} \qquad \cosh x = \frac{e^x + e^{-x}}{2}, \text{ hyperbolic cosine,}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \text{ hyperbolic tangent,}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \text{ hyperbolic cotangent.}$$

The hyperbolic cosine is an even function, i.e. $\cosh(-x) = \cosh x$, and hyperbolic sine, tangent and cotangent are odd functions: $\sinh(-x) = -\sinh x$, $\tanh(-x) = -\tanh x$, $\coth(-x) = -\coth x$.

It is useful to bear in mind that $\sinh 0 = 0$, $\cosh 0 = 1$, $\cosh^2 x - \sinh^2 x = 1$, $\tanh x \coth x = 1$.

The graphs of the hyperbolic functions $y = \sinh x$, $y = \cosh x$ and $y = \tanh x$ are shown, respectively, in Figs. 53-55.

The graph of the hyperbolic cosine is known as a *catenary*, which is a curve in which a heavy cable hangs when suspended from two points.

The derivatives of the hyperbolic functions can be found by the formulas

$$(\sinh x)' = \cosh x, \quad (\cosh x)' = \sinh x,$$
$$(\text{thanh } x)' = 1/\cosh^2 x, \quad (\coth x)' = -1/\sinh^2 x.$$

To integrate the hyperbolic functions, use is made of formulas

$$\int \sinh x\, dx = \cosh x + C, \quad \int \cosh x\, dx = \sinh x + C,$$

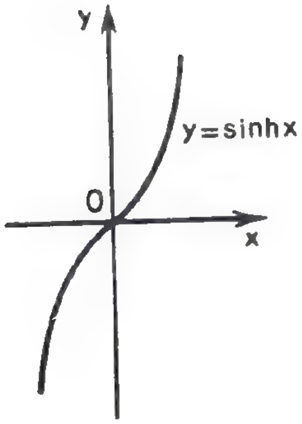

Fig. 53

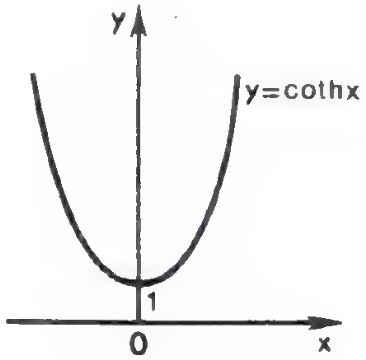

Fig. 54

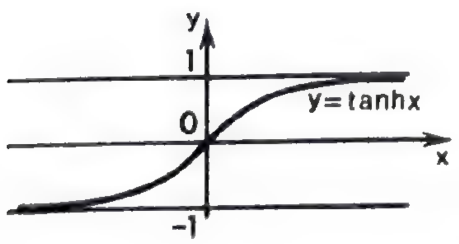

Fig. 55

$$\int \frac{dx}{\cosh^2 x} = \tanh x + C, \quad \int \frac{dx}{\sinh^2 x} = -\coth x + C.$$

1655. Prove the validity of the equality

$$\sinh (x + a) = \sinh x \cosh a + \cosh x \sinh a.$$

Solution. By the definition of the hyperbolic cosine, we have

$$\sinh (x + a) = \frac{e^{x+a} - e^{-(x+a)}}{2} = \frac{e^x \cdot e^a - e^{-x} \cdot e^{-a}}{2}.$$

Since $e^x = \cosh x + \sinh x$, $e^{-x} = \cosh x - \sinh x$, $e^a = \cosh a + \sinh a$, $e^{-a} = \cosh a - \sinh a$, we have

$$\sinh (x + a)$$
$$= \frac{(\cosh x + \sinh x)(\cosh a + \sinh a) - (\cosh x - \sinh x)(\cosh a - \sinh a)}{2}.$$

Perfoming algebraic transformations, we obtain

$$\sinh (x + a) = \sinh x \cosh a + \cosh x \sinh a.$$

1656. Express $\cosh (x + a)$ in terms of the hyperbolic functions of the arguments x and a.

Solution. Differentiating the equality

$$\sinh (x + a) = \sinh x \cosh a + \cosh x \sinh a$$

with respect to x, we obtain

$$\cosh (x + a) = \cosh x \cosh a + \sinh x \sinh a.$$

1657. Express $\sinh 2x$ and $\cosh 2x$ in terms of $\sinh x$ and $\cosh x$.

Solution. We have

$$\sinh 2x = \sinh (x + x) = \sinh x \cosh x + \cosh x \sinh x,$$
$$\cosh 2x = \cosh (x + x) = \cosh x \cosh x + \sinh x \sinh x,$$

i.e.

$$\sinh 2x = 2 \sinh x \cosh x; \cosh 2x = \cosh^2 x + \sinh^2 x.$$

1658. Express $\cosh^2 x$ and $\sinh^2 x$ in terms of $\cosh 2x$.

Solution. Solving the system of equations

$$\begin{cases} \cosh^2 x + \sinh^2 x = \cosh 2x, \\ \cosh^2 x - \sinh^2 x = 1 \end{cases}$$

with respect to $\cosh^2 x$ and $\sinh^2 x$, we get

$$\cosh^2 x = (\cosh 2x + 1)/2, \sinh^2 x = (\cosh 2x - 1)/2.$$

1659. Express the hyperbolic functions $\cosh xi$ and $\sinh xi$ of the imaginary argument in terms of $\sin x$ and $\cos x$.

Solution. We find

$$\sin xi = \frac{e^{xi} - e^{-xi}}{2} = i \cdot \frac{e^{xi} - e^{-xi}}{2i} = i \cdot \sin x,$$

$$\cosh xi = \frac{e^{xi} + e^{-xi}}{2} = \cos x.$$

Thus we have $\sinh xi = i \cdot \sinh x$, $\cosh xi = \cos x$.

1660. What line is specified by the parametric equations $x = a \cosh t$, $x = a \sinh t$ at $a > 0$?

Solution. Let us remove t from the equations, for which purpose we shall subtract y^2 from x^2:

$$x^2 - y^2 = a^2 (\cosh^2 t - \sinh^2 t), \quad \text{i.e.} \quad x^2 - y^2 = a^2.$$

The curve $x^2 - y^2 = a^2$ is an equilateral hyperbola whose asymptotes are the straight lines $y = \pm x$. The given curve is the right branch of the hyperbola since $x = a \cosh t > 0$ for any t (Fig. 56).

1661. Point M lies on the right branch of the equilateral hyperbola $x = a \cosh t$, $y = a \sinh t$. A perpendicular MN is dropped from the point M on the abscissa axis and the same point is connected by the line segment OM with the origin. A perpendicular AK is erected from the vertex A of the hyperbola till the intersection with the segment OM at the point K (Fig. 57). Prove that $|NM|:a = \sinh t$, $|ON|:a = \cosh t$, $|AK|:a = \tanh t$.

Solution. We have

$$|NM|:a = y : a = \sinh t, \quad |ON| : a = x : a = \cosh t,$$

$$|AK| : a = |NM| : |ON| = (|NM| : a)/(|ON| : a)$$
$$= \sinh t/\cosh t = \tanh t.$$

1662. Point M lies on the right branch of the equilateral hyperbola $x = a \cosh t$, $y = a \sinh t$. Calculate the area of the hyperbolic sector bounded by the branch of the hyperbola, the abscissa axis and the segment OM (Fig. 58).

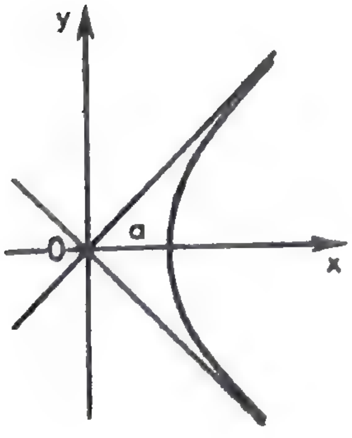

Fig. 56

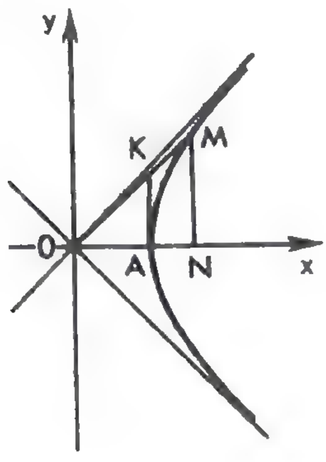

Fig. 57

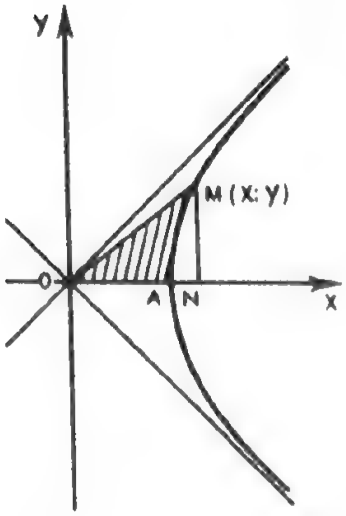

Fig. 58

Solution. We have $S = S_{ONM} - S_{ANM} = \frac{1}{2}xy - \int\limits_y^x y\,dx$. Since $x = a\cosh t$, $y = a\sinh t$, we have $dx = a\sinh t\,dt$, whence

$$S = \frac{1}{2}a^2 \sinh t \cdot \cosh t - a^2\int\limits_0^t \sinh^2 t\,dt = \frac{1}{2}a^2\sinh t\cdot\cosh t$$

$$-\frac{a^2}{2}\int\limits_0^t (\cosh 2t - 1)\,dt = \frac{1}{4}a^2\sinh 2t - \frac{1}{4}a^2\sinh 2t + \frac{a^2}{2}t = \frac{a^2 t}{2}.$$

Thus it follows that $t = 2S/a^2$. Consequently, the argument t of the hyperbolic functions can be regarded as the quotient resulting from the division of the doubled area of the hyperbolic sector OAM by the square of the real semi-axis.

1663. Find the derivatives of the following functions: (1) $y = \ln(\cosh x + \sqrt{\cosh^2 x + 1})$; (2) $y = 5\sinh^3(x/15) + 3\sinh^5(x/15)$; (3) $y = 2\arctan \times (\tanh(x/2))$; (4) $y = \tanh x - \frac{2}{3}\tanh^5 x + \frac{1}{5}\tanh^5 x$; (5) $y = \text{arccot}(1/\sinh x)$; (6) $y = \ln\tanh(x^2/4)$.

1664. At what point of the catenary $y = \cosh x$ do the tangent and the abscissa axis form the angle $\alpha = \pi/4$?

1665. Test the function $y = \cosh\frac{x}{2} - 1$ for an extremum.

1666. Find the following integrals: (1) $\int x^2\cosh x\,dx$; (2) $\int \sinh^4 x\,dx$; (3) $\int \frac{\tanh x}{\sqrt{\cosh x - 1}}dx$; (4) $\int \sinh x \sin x\,dx$; (5) $\int \frac{\sinh(x/2)}{\cosh^3(x/2)}dx$; (6) $\int \sinh^3(x/3)\cosh^2(x/3)\,dx$.

1667. Calculate the following definite integrals: (1) $\int\limits_0^{\ln(1+\sqrt{2})} \frac{\cosh x\, dx}{\sqrt{4-\sinh^2 x}}$;

(2) $\int\limits_0^{\ln 2} \tanh^2 x\, dx$; (3) $\int\limits_0^{\ln 3} x \cosh x\, dx$.

1668. Express $\sinh(x - a)$ and $\cosh(x - a)$ in terms of the hyperbolic functions of the arguments x and a.

1669. Express $\tanh(x + a)$ and $\tanh(x - a)$ in terms of $\tanh x$ and $\tanh a$. Find $\tanh 2x$.

1670. Express the functions $\sinh(x/2)$, $\cosh(x/2)$ and $\tanh(x/2)$ in terms of $\cosh x$.

1671. Reduce the expressions $\sinh x \pm \sinh y$, $\cosh x \pm \cosh y$, $\tanh x \pm \pm \tanh y$ to the form convenient for taking the logarithms.

1672. Express $\sinh x$ and $\cosh x$ in terms of $\tanh(x/2)$.

1673. Represent the products of the hyperbolic functions $\sinh x \cosh y$, $\sinh x \sinh y$, $\cosh x \cosh y$ in the form of a sum.

1674. Calculate the area bounded by the curve $y = \sinh x$ and the straight lines $x = \ln 5$, $y = 0$.

1675. Find the arc length of the curve $y = a\cosh(x/a)$ contained between the straight lines $x = 0$, $x = a$.

1676. Given two points M and N on the curve $x = a\cosh t$, $y = a\sinh t$, which correspond to the values $t = t_1$ and $t = t_2$ $(t_1 < t_2)$. Calculate the area of the sector OMN.

1677. What line is specified by the equations $x = a/\cosh t$, $y = b\tanh t$, if $a > 0$, $b > 0$?

1678. What line is specified by the equations $x = \cosh^2 t$, $y = \sinh^2 t$?

1679. Given $\sin\alpha = \tanh t$. Express $\cos\alpha$ and $\tan\alpha$ in terms of t.

1680. Simplify the expression

$$(\cos x \cosh y + i \sin x \sinh y)^2 - (\cos x \sinh y + i \sin x \cos y)^2.$$

1681. Simplify the expression $(x\cosh t + y\sinh t)^2 - (x\sinh t + y\cosh t)^2$.

1682. Prove the identities:

(1) $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$;

(2) $\cosh nx = \dfrac{(\cosh x + \sinh x)^n + (\cosh x - \sinh x)^n}{2}$;

(3) $\sinh nx = \dfrac{(\cosh x + \sinh x)^n - (\cosh x - \sinh x)^n}{2}$.

1683. Using the equalities $\sinh^n x = \dfrac{(e^x - e^{-x})^n}{2^n}$, $\cosh^n x = \dfrac{(e^x + e^{-x})^n}{2^n}$

prove that

$$\cosh^3 x = \frac{1}{4}\cosh 3x + \frac{3}{4}\cosh x, \quad \sinh^5 x = \frac{1}{16}\sinh 5x - \frac{5}{16} 3x + \frac{5}{8} \text{sinx } x.$$

Chapter 11

Elements of Linear Programming

11.1. Linear Inequalities and the Domain of Solutions of a System of Linear Inequalities

Suppose we are given a linear inequality with two variables x_1 and x_2:

$$a_1x_1 + a_2x_2 + b \geqslant 0. \tag{1}$$

If we consider x_1 and x_2 to be the coordinates of a point of a plane, then the set of the points of the plane whose coordinates satisfy inequality (1) is called the *domain of solutions* of the given inequality. The domain of solutions of inequality (1) is a half-plane.

To determine which of the two half-planes corresponds to inequality (1), it is sufficient to reduce the inequality to the form $x_2 \geqslant kx_1 + l$ or to the form $x_2 \leqslant kx_1 + l$. In the first case the desired half-plane lies above the straight line $a_1x_1 + a_2x_2 + b = 0$, in the second case, it lies below it. Now if $a_2 = 0$, the inequality can be reduced to one of the forms $x_1 \geqslant h$ or $x_1 \leqslant h$, that is, the half-plane lies on the right or on the left of the line $x_1 = h$.

If we are given the system of inequalities

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + b_1 \geqslant 0, \\ a_{21}x_1 + a_{22}x_2 + b_2 \geqslant 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + b_m \geqslant 0, \end{cases} \tag{2}$$

where m is a finite number, we obtain an intersection of a finite number of half-planes generating a polygonal domain D, which is called a *domain of solutions* of the system of inequalities (2). The domain is not always bounded; it can be unbounded or even empty. The last case occurs when the system of inequalities (2) is contradictory. Cases may also occur when there are superfluous inequalities which appear in a consistent system and specify the straight lines which have no points in common with the domain D. Such inequalities can be eliminated.

A domain of solutions possesses an important property of being *convex* which means that together with any two of its points it contains the whole line segment connecting them. A straight line possessing at least one point in common with the domain so that the whole domain lies on one side of this line is called the *line of support* relative to this domain.

A system of inequalities with three variables admits of similar geometrical interpretation:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + b_1 \geqslant 0, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + b_2 \geqslant 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + b_m \geqslant 0. \end{cases} \tag{3}$$

Each of the inequalities in this system is satisfied for one of the half-spaces into which the respective plane divides the whole space. The system of inequalities (3) is an intersection of half-spaces, that is, a polyhedral domain of solutions of a system of inequalities.

1684. Find the half-plane specified by the inequality $2x_1 + 3x_2 - 12 \leqslant 0$.
Solution. Replacing the inequality sign by the sign of strict equality, we get the equation of the straight line $2x_1 + 3x_2 - 12 = 0$, or $x_2 = (-2/3)x_1 + 4$ (Fig. 59).

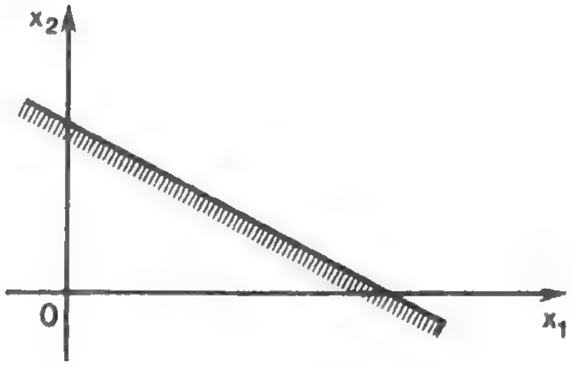

Fig. 59

Reducing the given inequality to the form $x_2 \leqslant (-2/3)x_1 + 4$, we see that the sought-for half-plane lies below the straight line $x_2 = (-2/3)x_1 + 4$.

1685. What half-plane is specified by the inequality $2x_1 - 3x_2 \geqslant 0$?
Solution. Replacing the inequality sign by that of strict equality, we get the equation of the straight line $2x_1 - 3x_2 = 0$, or $x_2 = (2/3)x_1$ passing through the origin. It follows from the inequality $2x_1 - 3x_2 \geqslant 0$, or $x_2 \leqslant (2/3)x_1$, that the desired half-plane lies below the line $x_2 = (2/3)x_1$ (Fig. 60).

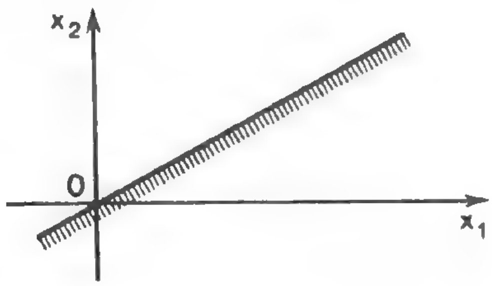

Fig. 60

1686. Find the domain of solutions of the system of inequalities $x_1 - 1 \geqslant 0$, $x_2 - 1 \geqslant 0$, $x_1 + x_2 - 3 \geqslant 0$, $-6x_1 - 7x_2 + 42 \geqslant 0$.
Solution. Replacing the inequality signs by the signs of strict equalities, we get the equations of four straight lines: $x_1 - 1 = 0$, $x_2 - 1 = 0$, $x_1 + x_2 - 3 = 0$ and $6x_1 + 7x_2 - 42 = 0$, shown in Fig. 61. Let us reduce the given inequalities to the form $x_1 \geqslant 1$, $x_2 \geqslant 1$, $x_2 \geqslant -x_1 + 3$, $x_2 \leqslant (-6/7)x_1 + 6$. The half-planes serving as the domains of sulutions of the respective inequalities are shown hatched. The domain of solutions of the system of inequalities is a convex tetrahedron.

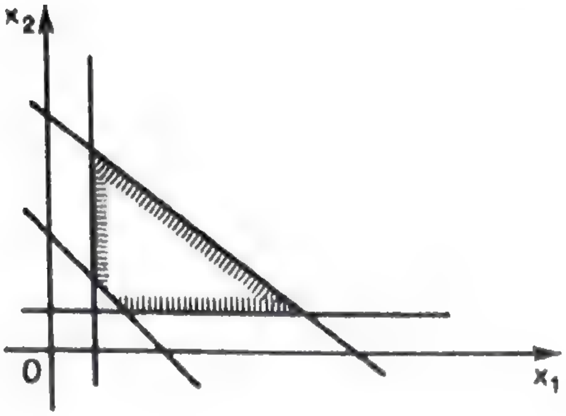

Fig. 61

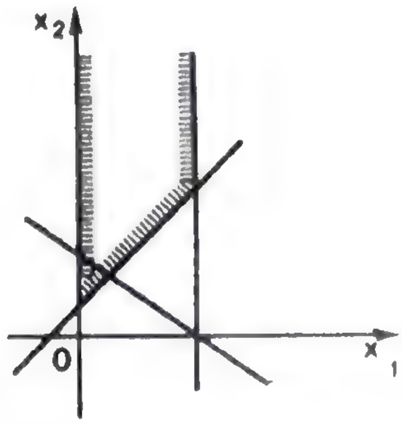

Fig. 62

1687. Find the domain of solutions of the system of inequalities $x_1 \geqslant 0$, $x_1 + x_2 - 2 \geqslant 0$, $x_1 - x_2 + 1 \leqslant 0$, $x_1 \leqslant 2$.

Solution. Replacing the signs of inequalities by those of strict equalities, we get the equations of four straight lines: $x_1 = 0$, $x_1 + x_2 - 2 = 0$, $x_1 - x_2 + 1 = 0$, $x_1 = 2$, shown in Fig. 62. Let us reduce the given inequalities to the form $x_1 \geqslant 0$, $x_2 \geqslant -x_1 + 2$, $x_2 \leqslant x_1 + 1$, $x_1 \leqslant 2$. The domain of solutions of the system of inequalities is an unbounded convex figure.

1688. Find the domain of solutions of the system of inequalities $x_1 \geqslant 2$, $x_1 + 3x_2 \leqslant 3$, $x_1 - x_2 + 1 \leqslant 0$.

Solution. Let us construct the corresponding straight lines. Figure 63 demonstrates that there is no point common to all the three half-planes. This means that the domain of solutions is empty and the given system of inequalities is inconsistent.

1689. Find the domain of solutions of the system of inequalities $2x_1 - x_2 \geqslant -2$, $x_1 - x_2 \geqslant -2$, $x_1 \leqslant 1$, $2x_1 - x_2 \leqslant 3$.

Solution. This system has no solutions. In terms of geometry, this means that there is no point whose coordinates comply with all the inequalities of the system (Fig. 64).

1690. Find the domain of solutions of the system of inequalities $3x_1 - x_2 \geqslant 0$ (a), $x_1 - x_2 \leqslant 0$ (b), $2x_1 + x_2 \leqslant 6$ (c), $x_1 \leqslant 2$ (d), $3x_1 - x_2 \geqslant -4$ (e).

Solution. The five given inequalities are associated with the set of points of the

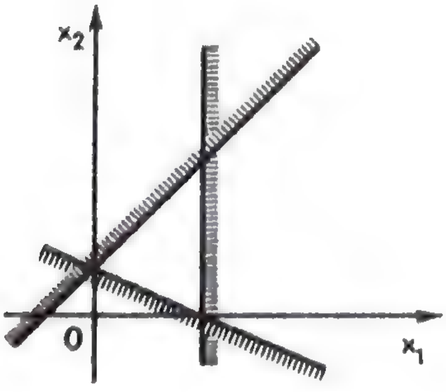

Fig. 63

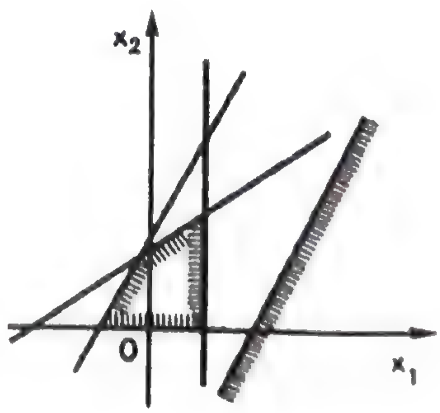

Fig. 64

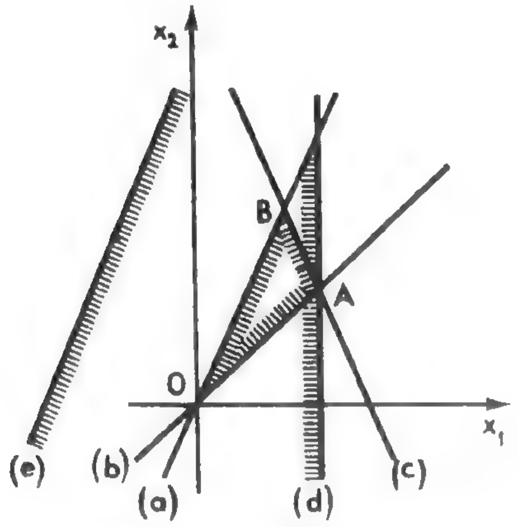

Fig. 65

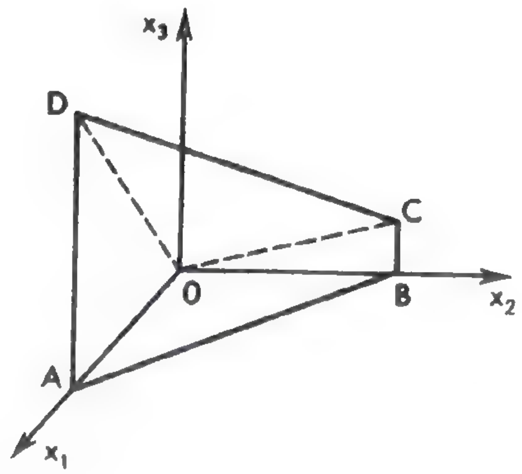

Fig. 66

plane forming the triangle AOB (Fig. 65). The inequalities (d) and (e) can be eliminated, since inequality (e) specifies the boundary line having no common points with the triangle AOB and the line specified by inequality (d) possesses one point in common with the triangle and is the line of support.

1691. Find the domain of solutions of the system of inequalities $x_1 \geqslant 0, x_2 \geqslant 0, x_3 \geqslant 0, x_1 + x_2 - 1 \leqslant 0, 3x_1 + x_2 - 3x_3 \geqslant 0$.

Solution. Replacing the inequality signs by the signs of strict equalities, we get the equations of the planes $x_1 = 0, x_2 = 0, x_3 = 0, x_1 + x_2 - 1 = 0, 3x_1 + x_2 - 3x_3 = 0$, which are represented in Fig. 66. The domain of solutions of the system of inequalities is the convex tetrahedron $ABOCD$.

1692. What is the position of the half-plane the coordinates of whose points satisfy the inequality $x_1 - x_2 - 10 \geqslant 0$?

1693. Find the domain of solutions of the system of inequalities $x_1 + x_2 - 5 \geqslant 0, x_1 - x_2 - 5 \geqslant 0, x_1 \leqslant 7$.

1694. Find the domain of solutions of the system of inequalities $x_1 - 5x_2 + 5 \geqslant 0, x_1 + 3x_2 - 3 \leqslant 0, x_1 \leqslant 5$.

1695. Find the domain of solutions of the system of inequalities $x_1 \geqslant 3, x_2 \geqslant 0, x_1 + x_2 \leqslant 0$.

1696. Find the domain of solutions of the system of inequalities $x_1 - x_2 + 1 \geqslant 0, 2x_1 + x_2 - 7 \geqslant 0, x_1 - 2x_2 + 4 \geqslant 0$.

1697. Find the domain of solutions of the system of inequalities $x_2 \geqslant 0$ (a), $4x_1 - x_2 \geqslant 0$ (b), $x_2 \leqslant 6$ (c), $4x_1 + x_2 \leqslant 40$ (d), $x_1 - x_2 + 8 \geqslant 0$ (e).

1698. Find the domain of solutions of the system of inequalities $x_1 \geqslant 0, x_2 \geqslant 1, x_3 \geqslant 0, x_1 + x_2 + x_3 - 5 \leqslant 0$.

1699. Find the domain of solutions of the system of inequalities $x_1 \leqslant 4, 2x_2 - x_3 \geqslant 0, x_2 + x_3 \leqslant 3, x_1 \geqslant 0, x_3 \geqslant 0$.

11.2. The Main Problem of Linear Programming

The main problem of linear programming is the minimization or the maximization of a linear function subject to linear constraints.

The function whose greatest or least value is being sought is called an *objective function* and the collection of the values of the variables at which the greatest or the least value is attained defines the so-called *optimal plan*. Any other collection of values complying with the restrictions defines the *feasible plan* (*solution*).

Assume that the constraints are given as a consistent system of m linear inequalities in n variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geqslant b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \geqslant b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \geqslant b_m. \end{cases}$$

It is required to choose among the nonnegative solutions of the system a solution as a result of which the linear (objective) function

$$L = c_1x_1 + c_2x_2 + \ldots + c_nx_n + c_0$$

assumes the greatest (the least) value or, as they say, it is required to maximize (minimize) the linear form L.

We shall show now how to solve this problem by a geometrical method for which purpose we shall consider a consistent system of linear inequalities in two and three variables. Suppose we are given, in addition, the linear function $L = c_1x_1 + c_2x_2 + c_0$. Let us choose from among the set of points $(x_1; x_2)$ belonging to the domain of solutions of the consistent system of inequalities such points which impart to the given linear function the least (the greatest) value. For each point of the plane the function L assumes a fixed value $L = L_1$. The set of all such points is the straight line $c_1x_1 + c_2x_2 + c_0 = L_1$ perpendicular to the vector $\mathbf{C}(c_1; c_2)$ which starts at the origin. If we move this line parallel to itself in the positive direction of the vector $\mathbf{C}$, then the linear function $L = c_1x_1 + c_2x_2 + c_0$ will increase, and if we move the line in the opposite direction, it will decrease. Suppose that when the line L moves in the positive direction of the vector $\mathbf{C}$, it first comes across the polygon of the solutions at its vertex and then, being in the position L_1, the line L becomes *supporting*, and on this line the function L assumes the least value. In its further movement in the same (positive) direction, the straight line L will pass through another vertex of the polygon of solutions, leaving the domain of solutions, and will also become a supporting line L_2; on that line, the function L assumes the greatest value among all the values attained on the polygon of solutions.

Thus, minimization and maximization of the linear function $L = c_1x_1 + c_2x_2 + c_0$ on the polygon of solutions are attained at the points of intersection of the polygon and the lines of support perpendicular to the vector $\mathbf{C}(c_1; c_2)$. The line of support may have either one point in common with the polygon of solutions (the vertex of the polygon), or an infinite set of points (this set being a side of the polygon).

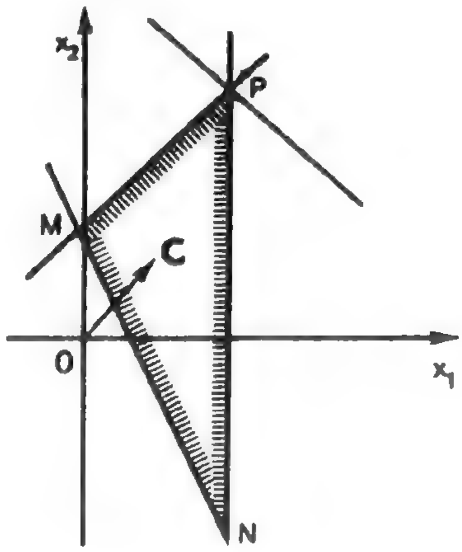

Fig. 67

By analogy with the aforesaid, the linear function of three variables $L = c_1x_1 + c_2x_2 + c_3x_3 + c_0$ assumes a constant value on the plane perpendicular to the vector $\mathbf{C}(c_1; c_2; c_3)$. The least and the greatest value of the function on the polyhedron of solutions are attained at the points of intersection of the polyhedron and the planes of support perpendicular to the vector $\mathbf{C}(c_1; c_2; c_3)$. A plane of support may have either one point in common with the polyhedron of solutions (its vertex), or an infinite set of points (the set being an edge or a face of the polyhedron).

1700. Maximize the linear form $L = 2x_1 + 2x_2$ subject to the constraints $3x_1 - 2x_2 \geqslant -6$, $3x_1 + x_2 \geqslant 3$, $x_1 \leqslant 3$.

Solution. Replacing the inequality signs by the signs of strict equalities, we construct the domain of solutions from the equations of the straight lines $3x_1 - 2x_2 + 6 = 0$, $3x_1 + x_2 - 3 = 0$, $x_1 = 3$ (Fig. 67). The domain of solutions of the inequalities is the triangle MNP. Now we construct the vector $\mathbf{C}(2; 2)$. Then, when leaving the triangle of solutions, the line of support passes through the point $P(3; 15/2)$ and, therefore, at the point P the linear function $L = 2x_1 + 2x_2$ assumes the greatest value, that is, is maximized, and $L_{max} = 2 \cdot 3 + 2 \cdot (15/2) = 21$.

1701. Minimize the linear function $L = 12x_1 + 4x_2$ subject to the constraints $x_1 + x_2 \geqslant 2$, $x_1 \geqslant 1/2$, $x_2 \leqslant 4$, $x_1 - x_2 \leqslant 0$.

Solution. Replacing the inequality signs by the signs of strict equalities, we construct the domain of solutions bounded by the straight lines $x_1 + x_2 = 2$, $x_1 = 1/2$, $x_2 = 4$, $x_1 - x_2 = 0$. The domain of solutions is the polygon $MNPQ$ (Fig. 68). Next we construct the vector $\mathbf{C}(12; 4)$. The line of support passes through the point $M(1/2; 3/2)$, which is the first point of intersection of the polygon of solutions and the line L on the way of the movement of that line in the positive direction of the vector $\mathbf{C}$. At the point M the linear function $L = 12x_1 + 4x_2$ assumes the least value $L_{min} = 12 \cdot (1/2) + 4 \cdot (3/2) = 12$.

1702. Find the greatest value of the function $L = x_1 + 3x_2 + 3x_3$ subject to the constraints $x_2 + x_3 \leqslant 3$, $x_1 - x_2 \geqslant 0$, $x_2 \geqslant 1$, $3x_1 + x_2 \leqslant 15$.

Solution. Let us construct the domain of solutions of the system of inequalities

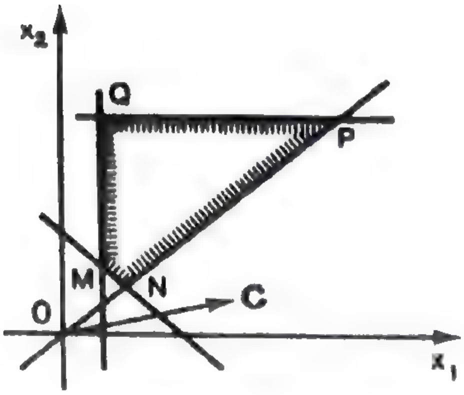

Fig. 68

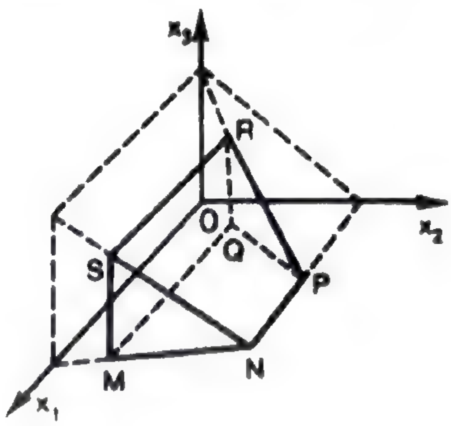

Fig. 69

from the equations of the planes $x_2 + x_3 = 3$, $x_1 - x_2 = 0$, $x_2 = 1$, $3x_1 + x_2 = 15$. The domain of solutions is the polyhedron $MNPQRS$ (Fig. 69).

We construct the vector $\mathbf{C}(1;\ 3;\ 3)$. When the plane of support moves in the positive direction of the vector $\mathbf{C}$, it leaves the polyhedron of solutions at the point $N(4;\ 3;\ 0)$. Therefore, at the point N the linear function $L = x_1 + 3x_2 + 3x_3$ assumes the greatest value, i.e. $L_{max} = 4 + 3 \cdot 3 + 3 \cdot 0 = 13$.

1703. Find the greatest value of the function $L = 3x_1 - 6x_2 + 2x_3$ subject to the constraints $3x_1 + 3x_2 + 2x_3 \leqslant 6$, $x_1 + 4x_2 + 8x_3 \leqslant 8$.

Solution. We construct the domain of solutions of the system of linear inequalities taking the planes $3x_1 + 3x_2 + 2x_3 = 6$, $x_1 + 4x_2 + 8x_3 = 8$, $x_1 = 0$, $x_2 = 0$, $x_3 = 0$. The domain is the polyhedron $MNOPR$ (Fig. 70). Now we construct the vector $\mathbf{C}(3;\ -6;\ 2)$. When the plane of support moves in the positive direction of the vector $\mathbf{C}$, it leaves the polyhedron of solutions at the points of the edge MR. Consequently, the greatest value is attained by the given function at the points of the line segment MR. We can ascertain this fact by substituting the coordinates of the points $M(2;\ 0;\ 0)$ and $R(16/11;\ 0;\ 9/11)$ into the linear form L; we obtain $L(M) = 6$, $L(R) = 6$.

1704. Find the greatest value of the function $L = x_1 + 3x_2$ subject to the constraints $x_1 + 4x_2 \geqslant 4$, $x_1 + x_2 \leqslant 6$, $x_2 \leqslant 2$.

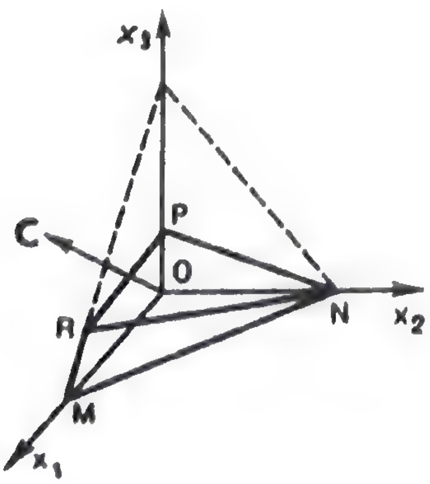

Fig. 70

1705. Minimize the function $L = x_1 - x_2$ subject to the constraints $3 \leqslant x_1 + x_2 \leqslant 7$, $1 \leqslant x_2 \leqslant 4$, $x_1 \leqslant 4$.

1706. Find the greatest value of the function $L = 3x_1 - 4x_2$ subject to the constraints $x_1 - 2x_2 \geqslant 6$, $x_1 + 2x_2 \geqslant 0$, $x_1 \leqslant 6$.

1707. Find the greatest value of the function $L = -x_1 + 2x_2$ subject to the constraints $x_1 - 8x_2 \leqslant 10$, $x_1 + x_2 \geqslant 1$, $x_1 - 5x_2 \geqslant -5$, $3x_1 + 10x_2 \leqslant 30$.

1708. Find the greatest value of the function $L = 8x_1 - 2x_2$ subject to the constraints $3x_1 + 4x_2 \geqslant 18$, $3x_1 - x_2 \geqslant 3$, $x_2 \leqslant 6$, $2x_1 + x_2 \leqslant 18$, $4x_1 - x_2 \leqslant 24$.

1709. Minimize the linear form $L = -2x_1 - x_2 + 3x_3$ subject to the constraints $x_1 + x_2 \geqslant 2$, $3x_1 + x_2 \leqslant 6$, $x_3 \leqslant 3$.

1710. Find the greatest value of the function $L = x_1 + 2x_2 + 3x_3$ subject to the constraints $x_1 + x_2 \leqslant 3$, $x_1 + x_2 - x_3 \leqslant 0$, $3x_1 + 3x_2 - x_3 \geqslant 0$.

1711. Find the greatest value of the function $L = 10x_1 + x_3$ subject to the constraints $3x_1 + 2x_2 + x_3 \leqslant 6$, $3x_1 - 3x_2 + x_3 \leqslant 6$, $x_3 \leqslant 3$.

11.3. The Simplex Method

11.3.1. The concept of the simplex method. The solution of the main problem of linear programming by the geometrical method is visual in the case of two or even three variables, but if the variables are more than three, the geometrical method becomes unfeasible. The so-called *simplex method* is an analytical method of solution of the main problem of linear programming. The system of constraints in calculus is usually given by the system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_2 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m, \end{cases} \qquad (1)$$

and the problem is to find, among the nonnegative solutions of the system of equations (1), such solutions that would maximize the linear function

$$L = c_1x_1 + c_2x_2 + \ldots + c_nx_n + c_0.$$

Let us express $x_1, x_2, \ldots, x_r$ $(r \leqslant m)$ in terms of the rest of the variables:

$$\begin{cases} x_1 = a'_{1,r+1}x_{r+1} + \ldots + a'_{1n}x_n + b'_1, \\ x_2 = a'_{2,r+1}x_{r+1} + \ldots + a'_{2n}x_n + b'_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_r = a'_{r,r+1}x_{r+1} + \ldots + a'_{rn}x_n + b'_r, \end{cases} \qquad (2)$$

where $b'_1 \geqslant 0$, $b'_2 \geqslant 0, \ldots, b'_r \geqslant 0$. If the constraints are given by inequalities, they can be transformed into equalities by introducing new nonnegative variables, the so-called *slack* variables. Thus, for instance, it is sufficient to add the quantity $x_{n+1} \geqslant 0$ to the left-hand side of the inequality $a_1x_1 + a_2x_2 + \ldots + a_nx_n \leqslant b$, to

get the equality

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n + x_{n+1} = b.$$

The constraints can also be mixed, that is, they can be inequalities and equations. Then the indicated method can be used to reduce them to equations alone. The variables (unknowns) $x_1, x_2, \ldots, x_r$ are called *basic*, and the whole collection $\{x_1, x_2, \ldots, x_r\}$ is called the *basis*, the rest of the variables being called *nonbasic* (*free*). The system of constraints (2) is called the *system reduced to a unit basis*. Replacing in the linear form L the basic variables by their expressions in terms of the nonbasic variables from system (2), we obtain

$$L = \gamma_0 + \gamma_{r+1}x_{r+1} + \ldots + \gamma_nx_n.$$

Assuming now all the nonbasic variables to be equal to zero, we find the values of the basic variables: $x_1 = b_1', x_2 = b_2', \ldots, x_r = b_r'$. Thus, the solution $(b_1', b_2', \ldots, b_r', 0, \ldots, 0)$ of the system is feasible, it is called the *basic* solution. For the basic solution to be obtained, the value of the linear form must be $L_B = \gamma_0$. A solution of a problem by means of the simplex method consists of several steps, ensuring a successive passage from the given basis B to another basis B$'$ with the aim of decreasing the value L_B, or at least not increasing it, i.e. $L_{B'} \leqslant L_B$.

The following concrete examples will provide for better understanding of the method.

1712. Maximize the linear form $L = -x_4 + x_5$ subject to the constraints $x_1 + x_4 - 2x_5 = 1$, $x_2 - 2x_4 + x_5 = 2$, $x_3 + 3x_4 + x_5 = 3$.

Solution. The given system of equations-constraints is consistent since the ranks of the matrix of the system

$$\begin{pmatrix} 1 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 3 & 1 \end{pmatrix}$$

and of the augmented matrix

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & -2 & 1 & 2 \\ 0 & 0 & 1 & 3 & 1 & 3 \end{array}\right)$$

coincide and are equal to 3. Consequently, the system of equations is consistent and three (basic) variables can be linearly expressed in terms of two nonbasic variables. Let us express, for example, x_1, x_2 and x_3 in terms of x_4 and x_5, that is, reduce the system to the unit basis:

$$\begin{cases} x_1 = 1 - x_4 + 2x_5, \\ x_2 = 2 + 2x_4 - 2x_5, \\ x_3 = 3 - 3x_4 - 2x_5. \end{cases} \qquad (*)$$

We express the linear form $L = -x_4 + x_5$ in terms of the nonbasic variables x_4 and x_5 (in the given example L has already been expressed in terms of x_4 and x_5). Having now $x_4 = 0$ and $x_5 = 0$, we find the values of the basic variables: $x_1 = 1$, $x_2 = 2, x_3 = 3$. Thus, the first feasible solution of the system of equations is $x_1 = 1$, $x_2 = 2, x_3 = 3, x_4 = 0, x_5 = 0$ or $(1, 2, 3, 0, 0)$. With the feasible solution obtained, the linear form L is of the value 0, i.e. $L_1 = 0$.

We shall now try to increase the value of L_1; an increase in x_4 will bring a decrease in L_1 since x_4 is preceded by a negative coefficient, and an increase in x_5 brings about an increase in L_1. So we shall increase x_5, without making x_1, x_2, x_3 negative, leaving $x_4 = 0$. It follows from the second equation of system (*) that x_5 can be increased to 2. Thus we get the following values of the variables: $x_1 = 5, x_2 = 0, x_3 = 1, x_4 = 0, x_5 = 2$, or $(5, 0, 1, 0, 2)$.

With the feasible solution, the value of the linear form L is $L_2 = 2$. After the second step the value of L has increased.

Next we shall take x_2 and x_4 to be nonbasic variables, that is, we shall take the variables which are zero in the new solution. With this aim in view, we shall express x_5 in terms of x_2 and x_4 in the second equation of system (*) and obtain $x_5 = 2 - x_2 + 2x_4$. Then

$$\left\{\begin{aligned} x_1 &= 5 - 2x_2 + 3x_4, \\ x_3 &= 1 + x_2 - 5x_4, \\ x_5 &= 2 - x_2 + 2x_4, \end{aligned}\right. \qquad (**)$$
$$L = 2 - x_2 + 2x_4.$$

To increase the value of L, we shall increase x_4. It can be seen in the second equation of system (* *) that if x_3 is nonnegative, the value of x_4 can be made $x_4 = 1/5$. Under this condition, the new feasible solution is $x_1 = 28/5, x_2 = 0, x_3 = 0, x_4 = 1/5, x_5 = 12/5$, or $(28/5, 0, 0, 1/5, 12/5)$. This time the value of the linear form is $L_3 = 11/5$.

Now we shall express x_1, x_4, x_5 in terms of the nonbasic variables x_2 and x_3:

$$\left\{\begin{aligned} x_1 &= 28/5 - (3/5)x_3 - (7/5)x_2, \\ x_4 &= 1/5 - (1/5)x_3 + (1/5)x_2, \\ x_5 &= 12/5 - (2/5)x_3 - 3/5x_2, \end{aligned}\right. \qquad (***)$$
$$L = 11/5 - (1/5)x_3 - (4/5)x_2.$$

Since in the last linear form both nonbasic variables are with negative coefficients, the greatest value is attained by L at $x_2 = 0, x_3 = 0$. This means that the solution $(28/5, 0, 0, 1/5, 12/5)$ is optimal and $L_{max} = 11/5$.

1713. Maximize the linear form $L = x_2 + x_3$ subject to the constraints $x_1 - x_2 + x_3 = 1, x_2 - 2x_3 + x_4 = 2$.

Solution. The system of equations-constraints is consistent since the ranks of the matrix of the system of equations and of the augmented matrix are the same and are equal to 2. Consequently, two basic variables can be linearly expressed in terms of the other two, nonbasic, variables. We shall take x_2 and x_3 as nonbasic variables.

Then we shall have

$$\begin{cases} x_1 = 1 + x_2 - x_3, \\ x_4 = 2 - x_2 + 2x_3 \end{cases}$$
$$L = x_2 + x_3.$$

At $x_2 = 0$, and $x_3 = 0$, the basic variables $x_1 = 1, x_4 = 2$, that is, we have the first feasible solution (1, 0, 0, 2) and $L_1 = 0$. An increase in L can be attained by increasing x_3 to the value 1. Then at $x_3 = 1$, $x_2 = 0$, the values of the basic variables are $x_1 = 0$, $x_4 = 0$. The new feasible solution is (0, 0, 1, 4) and $L_2 = 1$.

Let us now express x_3 and x_4 in terms of x_1 and x_2:

$$\begin{cases} x_3 = 1 - x_1 + x_2, \\ x_4 = 4 - 2x_1 + x_2, \end{cases}$$
$$L = 1 - x_1 + 2x_2.$$

An increase in L is possible with an increase in x_2, and the increase in x_2 is unlimited judging by the last system of equations. Thus, L will be assuming ever greater positive values, i.e. $L_{max} = +\infty$.

We see that the form L is not bounded above and, therefore, there is no optimal solution.

1714. Given the system of constraints $x_1 + x_2 + 2x_3 - x_4 = 3$, $x_2 + 2x_4 = 1$ and the linear form $L = 5x_1 - x_3$. Find the optimal solution minimizing the linear form.

Solution. This problem could be reduced to that of finding the maximum of the function $L_1 = -L$, i.e. $L_1 = -5x_1 + x_3$, but it is not necessary to do this. Reasoning as above, the problem can be solved without reducing it to maximization. The given system of equations is consistent since the ranks of the matrix of the system and of the augmented matrix are the same and are equal to 2. Consequently, the system of equations can be rewritten like this:

$$\begin{cases} x_1 = 2 - 2x_3 + 3x_4, \\ x_2 = 1 - 2x_4, \end{cases}$$
$$L = 10 - 11x_3 + 15x_4.$$

Here x_1 and x_2 serve as the basic variables and x_3 and x_4, as nonbasic variables. For $x_3 = 0$ and $x_4 = 0$ the first basic solution is $x_1 = 2, x_2 = 1, x_3 = 0, x_4 = 0$, or (2, 1, 0, 0) and $L_1 = 10$. A decrease in the linear form L is brought about by an increase in x_3 since x_3 in the form L is preceded by a negative coefficient, with an increase in x_3 being possible only up to 1, while the value $x_4 = 0$ remains unchanged. If we assume $x_3 = 1$, then $x_1 = 0, x_2 = 1, x_3 = 1, x_4 = 0$, or (0, 1, 1, 0) is the second basic solution for which $L_2 = -1$.

Let us express x_2 and x_3 in terms of the new nonbasic variables x_1 and x_4:

$$\begin{cases} x_2 = 1 - 2x_4, \\ x_3 = 1 - (1/2)x_1 + (3/2)x_4, \end{cases}$$
$$L = -1 + (11/2)x_1 - (3/2)x_4.$$

Now a decrease in the value of the form L depends on an increase in x_4 to the value $1/2$ (x_2 being nonnegative), while the value $x_1 = 0$ remains unchanged. In this case we have a new feasible solution $x_1 = 0, x_2 = 0, x_3 = 7/4, x_4 = 1/2$, or $(0, 0, 7/4, 1/2)$, for which $L = -7/4$.

Let us express x_3 and x_4 in terms of the nonbasic variables x_1 and x_2:

$$\begin{cases} x_3 = 7/4 + (1/2)x_1 - (3/4)x_2, \\ x_4 = 1/2 - (1/2)x_2, \end{cases}$$
$$L = -7/4 + (11/2)x_1 + (3/4)x_2.$$

Since any further decrease in the value of the form L is impossible because of the positive coefficients in x_1 and x_2, the feasible solution $(0, 0, 7/4, 1/2)$ of the problem is optimal. The least value of L is $-7/4$.

1715. Maximize the linear form $L = 2x_1 - x_4$ subject to the following constraints:

$$\begin{cases} x_1 + x_2 + 5x_3 = 20, \\ x_2 + 2x_4 \geqslant 5, \\ -x_1 - x_2 + x_3 \leqslant 8. \end{cases}$$

Solution. Since the given system of constraints is mixed, we shall reduce it to the system of equations by introducing a new nonnegative variable x_5 into the left-hand side of the second condition with a negative coefficient and x_6 into the third condition with a positive coefficient. Then we get the system of equations

$$\begin{cases} x_1 + x_2 + 5x_3 = 20, \\ x_2 + 2x_4 - x_5 = 5, \\ -x_1 - x_2 + x_3 + x_6 = 8. \end{cases}$$

Let us reduce the system to the unit basis taking x_1, x_2, x_3 as the basic variables (since the rank of the matrix of the system is equal to 3):

$$x_1 = 15 + 2x_4 - x_5, \quad x_2 = 5 - 2x_4 + x_5, \quad x_3 = 28 - x_6. \qquad (*)$$

Then the linear form assumes the form $L = 30 + 3x_4 - 2x_5$. For $x_4 = 0, x_5 = 0, x_6 = 0$, the basic variables have the values $x_1 = 15, x_2 = 5, x_3 = 28$, that is the first feasible solution is $(15, 5, 28, 0, 0, 0)$; in this case $L_1 = 30$.

To increase the value of L, it is necessary to increase x_4, since this variable appears in the expression for L with a positive coefficient. As can be seen from the second condition of the system of constraints $(*)$, an increase in x_4 is possible up to

the value 5/2. At $x_4 = 5/2$, $x_5 = 0$, $x_6 = 0$, the values of the other variables are the following: $x_1 = 20$, $x_2 = 0$, $x_3 = 28$, that is, the second feasible solution is (20, 0, 28, 5/2, 0, 0) and the linear function L assumes the form $L = 75/2 - (3/2)x_2 - - (1/2)x_5$, and with the second feasible solution its value is $L_2 = 75/2$.

Now when the coefficients in the variables in L are negative, an increase in the value of L is impossible. Consequently, $L_{max} = 75/2 = 37.5$.

1716. There is an amount of 100 kg of metal to manufacture two kinds of articles. One article of the first kind requires 2 kg of metal, and one article of the second kind, 4 kg. Compile a manufacturing plan providing for the maximum gain from the sale of the articles if the selling price of one article of the first kind is 3 rubles and that of one article of the second kind is 2 rubles, under the condition that it is required to manufacture not more than 40 articles of the first kind and not more than 20 articles of the second kind.

Solution. Assume that they have manufactured x_1 articles of the first kind and x_2 articles of the second kind. Then we have the following constraints imposed on the variables x_1 and x_2:

$$\begin{cases} x_1 \leqslant 40, \\ x_2 \leqslant 20, \\ 2x_1 + 4x_2 = 100. \end{cases}$$

The objective function has the form $L = 3x_1 + 2x_2$. We transform the mixed system of constraints into a system of constraints in the form of equations by introducing new variables x_3 and x_4:

$$\begin{cases} x_1 + x_3 = 40, \\ x_2 + x_4 = 20, \\ x_1 + 2x_2 = 50. \end{cases}$$

The rank of the matrix of the system

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & 0 \end{pmatrix}$$

is 3. Let us take x_1, x_2, x_3 as the basic variables and pass to the unit basis:

$$\begin{cases} x_1 = 10 + 2x_4, \\ x_2 = 20 - x_4, \\ x_3 = 30 - 2x_4. \end{cases}$$

The first feasible solution is obtained at $x_4 = 0$, $x_1 = 10$, $x_2 = 20$, $x_3 = 30$. For these values of the variables $L = 70$. An increase in the value of the objective function can be attained by increasing x_4 to the value 15, as can be judged by the third equation. Then for $x_4 = 15$, $x_1 = 40$, $x_2 = 5$, $x_3 = 0$ we have $L = 130$. The second

feasible solution is (40, 5, 0, 15); $x_1 = 40 - x_3$, $x_2 = 5 + (1/2)x_3$, $x_4 = 15 - (1/2)x_3$, $L = 130 - 2x_3$. In the objective function, the coefficient in x_3 is negative and so any further increase in L is impossible. Consequently, the optimal solution is $x_1 = 40$, $x_2 = 5$ and $L_{\max} = 130$.

11.3.2. Simplex tables. We reduce the system of constraints to the unit basis

$$x_1 + \ldots + a_{1,r+1}x_{r+1} + \ldots + a_{1n}x_n = b_1,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_i + \ldots + a_{i,r+1}x_{r+1} + \ldots + a_{in}x_n = b_i,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_r + \ldots + a_{r,r+1}x_{r+1} + \ldots + a_{rn}x_n = b_r,$$

and the linear function L to the form

$$L + \gamma_{r+1}x_{r+1} + \ldots + \gamma_j x_j + \ldots + \gamma_n x_n = \gamma_0 . \qquad (1)$$

These data can be tabulated as follows:

Basic variables	Nonbasic variables	x_1	...	x_i	...	x_r	x_{r+1}	...	x_j	...	x_n
x_1	b_1	1	...	0	...	0	$a_{1,r+1}$	...	a_{1j}	...	a_{1n}
.....	.....	.	.	.	...	...	.....	...	...	...	...
x_i	b_i	0	...	1	...	0	$a_{i,r+1}$	...	a_{ij}	...	a_{in}
.....	.....	.	.	.	...	...	.....	...	...	...	...
x_r	b_r	0	...	0	...	1	$a_{r,r+1}$	...	a_{rj}	...	a_{rn}
L	γ_0	0	...	0	...	0	γ_{r+1}	...	γ_j	...	γ_n

Equality (1) will be called a *reduced* (to nonbasic variables) *expression* for the function L, and the coefficients γ_j will be called *estimations* (indices) of the respective nonbasic variables x_j.

1. A resolvent column a_p is chosen from the condition: estimation $\gamma_p < 0$ and at least one element $a_{ip} > 0$.

2. The qth resolvent row is chosen from the condition

$$b_q/a_{qp} = \min\{b_i/a_{ip}\} \quad \text{for} \quad a_{ip} > 0.$$

3. The recalculation of the elements of the resolvent qth row is performed by the formula

$$a'_{qk} = a_{qk}/a_{qp} \quad (k = 0, 1, \ldots, n).$$

4. The elements of the remaining rows are calculated (for $k \neq p$) by the formula

$$a'_{ik} = a_{ik} - a'_{qk} \cdot a_{ip} \quad (i = 0, 1, \ldots, q-1, q+1, \ldots, r).$$

It pays to bear in mind the fundamental theorem of the simplex method which we give without proof.

Theorem. *If after some consecutive operation*

(1) *there is at least one negative estimation and each columm including such an estimation contains at least one positive element, i.e.* $\gamma_k > 0$ *for certain* k' s *and* $a_{ik} > 0$ *for the same* k' s *and for a certain* i, *then the solution can be improved by means of the next iteration;*

(2) *there is at least one negative estimation whose column does not include positive elements,* i.e. $\gamma_k < 0$, $a_{ik} < 0$, *for a certain* k *and for all* i, *then the function* L *is not bounded in the feasible domain* ($L_{\max} \to \infty$);

(3) *all estimations turn out to be nonnegative, i.e.* $\gamma_k \geqslant 0$ *for all* k, *then the optimal solution has been attained.*

1717. Find the least value of the linear function $L = 7x_1 + 5x_2$ on the set of nonnegative solutions of the system of equations

$$\begin{cases} 2x_1 + 3x_2 + x_3 = 19, \\ 2x_1 + x_2 + x_4 = 13, \\ 3x_2 + x_5 = 15, \\ 3x_1 + x_6 = 18. \end{cases}$$

Solution. The rank of the matrix of the system of equations

$$\begin{pmatrix} 2 & 3 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is 4. The rank of the augmented matrix is also 4. Consequently, four (basic) variables can be expressed in terms of two (nonbasic) variables, that is,

$$\begin{cases} x_3 = 19 - 2x_1 - 3x_2, \\ x_4 = 13 - 2x_1 - x_2, \\ x_5 = 15 - 3x_2, \\ x_6 = 18 - 3x_1. \end{cases}$$

Incidentally, the linear form $L = 7x_1 + 5x_2$, or $L - 7x_1 - 5x_2 = 0$, has been expressed in terms of these nonbasic variables. We have the following initial table (Table 1).

Table 1

Basic variables	Nonbasic variables	x_1	x_2	x_3	x_4	x_5	x_6
x_3	19	2	3	1	0	0	0
x_4	13	2	1	0	1	0	0
x_5	15	0	3	0	0	1	0
x_6	18	3	0	0	0	0	1
L	0	-7	-5	0	0	0	0

We verify whether there are negative estimations in the last (index) row and see that there are two such numbers: -7 and -5. We take -5, for example, and look through the column of x_2. There are three positive elements in the column: 3, 1, and 3. We divide the corresponding nonbasic variables by these numbers: 19/3, 13/1, 15/3. From the quotients obtained the least one is 15/3. Consequently, the resolvent element is 3 located at the intersection of the row of x_5 and the column of x_2. We separate out this row and this column by frames. The new basis consists of x_3, x_4, x_2, x_6. To compile the next table, we multiply the framed row of Table 1 by 1/3 in order to get 1 in the place of the resolvent element, and write the row thus obtained in the place of the previous one. To each of the remaining rows we add the newly obtained row multiplied by a number that will make zeros appear in the column of x_2, and write the transformed rows in the place of the old ones. Here ends the first iteration.

Table 2

Basic variables	Nonbasic variables	x_1	x_2	x_3	x_4	x_5	x_6
x_3	4	2	0	1	0	-1	0
x_4	8	2	0	0	1	$-1/3$	0
x_2	5	0	1	0	0	1/3	0
x_6	18	3	0	0	0	0	1
L	25	-7	0	0	0	5/3	0

Now the whole argument is repeated in reference to Table 2, that is, the second iteration is performed. The new resolvent element lying at the intersection of the row of x_3 and the column of x_1 is 2. We pass on to the next table.

The same reasoning is repeated in reference to Table 3. Here the resolvent element is 2/3 lying at the intersection of the row of x_4 and the column of x_5. We pass to Table 4. Since in the index row there are no negative estimations, we have obtained the optimal plan (5, 3, 0, 0, 6, 3) and the greatest value of the linear form L is $L_{max} = 50$.

Table 3

Basic variables	Nonbasic variables	x_1	x_2	x_3	x_4	x_5	x_6
x_1	2	1	0	1/2	0	−1/2	0
x_4	4	0	0	−1	1	2/3	0
x_2	5	0	1	0	0	1/3	0
x_6	12	0	0	−3/2	0	3/2	1
L	39	0	0	7/2	0	−11/6	0

Table 4

Basic variables	Nonbasic variables	x_1	x_2	x_3	x_4	x_5	x_6
x_1	5	1	0	−1/4	3/4	0	0
x_5	6	0	0	−3/2	3/2	1	0
x_2	3	0	1	1/2	−1/2	0	0
x_6	3	0	0	3/4	−9/4	0	1
L	50	0	0	3/4	11/4	0	0

Find the optimal nonnegative solutions minimizing the following linear forms:

1718. $\begin{cases} x_1 - 2x_2 + x_3 = 1, \\ x_1 + 3x_2 + x_4 = 2, \end{cases}$ $L = x_1 - x_3$.

1719. $\begin{cases} x_1 = 2 + 2x_3 - x_4, \\ x_2 = 1 + x_3 - 2x_4, \\ x_5 = 5 - x_3 + x_4, \end{cases}$ $L = x_1 + x_2$.

1720. $\begin{cases} 2x_1 + x_2 - x_3 - x_4 = 2, \\ x_2 - x_4 = 1, \\ L = 2x_3 - x_2. \end{cases}$

1721. $\begin{cases} 4x_1 + 3x_2 + x_3 = 180, \\ 4x_2 + 9x_3 + 12x_4 = 900, \end{cases}$ $L = 12x_1 + 5x_2 + 3x_3$.

1722. $\begin{cases} x_1 + x_4 + 6x_6 = 9, \\ 3x_1 + x_2 - 4x_3 + 2x_6 = 2, \\ x_1 + 2x_3 + x_5 + 2x_6 = 6, \end{cases}$ $L = x_1 - x_2 + x_3 + x_4 + x_5 - x_6$.

Find the optimal nonnegative solutions maximizing the following linear forms:

1723. $\begin{cases} x_1 + x_2 + 5x_3 = 20, \\ x_2 + 2x_4 \geqslant 5, \\ x_1 + x_2 - x_3 \geqslant 8, \end{cases}$ $L = 2x_1 + x_4.$

1724. $\begin{cases} x_1 - 2x_2 + 3x_3 \geqslant -1, \\ 2x_1 - x_2 - x_3 \leqslant -1, \end{cases}$ $L = -x_1 - 2x_2 - 3x_3.$

1725. The productive capacity of an assembly shop is 120 articles of type A and 360 articles of type B a day. Quality control capacity is 200 articles of either type a day. Articles of type A are four times as expensive as those of type B. It is required to plan the output so as to guarantee the factory the maximum profit.

1726. The storehouse can provide not more than 80 kg of metal for manufacturing articles of two types, articles of type I requiring 2 kg of metal and those of type II, 1 kg of metal. The problem is to plan the manufacturing process so as to ensure the maximum profit if it is required to produce not more than 30 pieces of type I and not more than 40 pieces of type II, one article of type I costing 5 rubles and of type II, 3 rubles.

1727. Two kinds of forage are required to feed the cattle. One kilogram of forage of the first kind costs 5 copecks and that of the second kind, 2 copecks. Each kilogram of forage of the first kind contains 5 units of nutrient A, 2.5 units of nutrient B and 1 unit of nutrient C, and each kilogram of forage of the second kind contains 3, 3 and 1.3 units respectively. What amount of forage of each kind should be spent every day to ensure the minimum expenditure for cattle feeding if the daily ration must include not less than 225 units of nutrient of type A, not less than 150 units of type B and not less than 80 units of type C?

11.3.3. The concept of a degenerate solution. When we considered the simplex method, we assumed that $b_i > 0$ (see p. 367) both in the initial system and in the systems obtained as a result of iterations. If in certain equations the nonbasic variables $b_i = 0$, then in the basic solution corresponding to that system the basic variables with respect to which the equations are solved assume zero values. The basic solution in which at least one of the basic variables takes on a zero value is called a *degenerate solution* and the problem of linear programming including at least one degenerate solution is called a *degenerate problem*. Applying succesive iterations in this case, we can return to the collection of basic and nonbasic variables we have come across earlier, that is, the so-called cycling occurs in the calculation scheme. Here is the rule of avoiding cycling (we do not give here any theoretical substantiation of the rule since it is a special question relating to the so-called degeneration problem).

Rule. *If an indeterminacy appears as to the selection of a resolvent row at some stage of calculations, that is, several equal minimum ratios b_i/a_{ip} turn out, the row should be selected for which the ratio of the elements of the column following the resolvent column to those of the resolvent column is the least. If it again results in an equal minimum ratios, then the ratio including the elements of the next column is compiled, and so on until the resolvent row is uniquely determined.*

1728. Maximize the linear form $L = 4x_5 + 2x_6$ subject to the constraints $x_1 + x_5 + x_6 = 12$, $x_2 + 5x_5 - x_6 = 30$, $x_3 + x_5 - 2x_6 = 6$, $2x_4 + 3x_5 - 2x_6 = 18$, $x_1 \geqslant 0$, $x_2 \geqslant 0$, $x_3 \geqslant 0$, $x_4 \geqslant 0$, $x_5 \geqslant 0$, $x_6 \geqslant 0$.

Solution. The basic solution and the objective value corresponding to the initial system are (12, 30, 6, 9, 0, 0) and $L = 0$ respectively. Given below is the successive iterations of the simplex method:

Initial Table

x_i	b_i					4	2	b_i/a_{ip}	a_{i1}/a_{ip}
		x_1	x_2	x_3	x_4	x_5	x_6		
x_1	12	1				1	1	12	
x_2	30		1			5	−1	6	−1/5
x_3	6			1		1	−2	6	−2
x_4	18				2	3	−2	6	−2/3
L	0					−4	−2		

Iteration 1

x_1	6	1		−1			3	2	
x_2	0		1	−5			9	0	−5/9
x_5	6			1		1	−2		
x_4	0			−3	2		4	0	−3/4
L	24			4			−10		

Iteration 2

x_1	6	1		5/4	−3/2			24/5	
x_2	0		1	7/4	−9/2			0	
x_5	6			−1/2	1	1			
x_6	0			−3/4	1/2				
L	24			−7/2	5				

Iteration 3

x_1	6	1	$-5/7$		$12/7$		
x_3	0		$4/7$	1	$-18/7$		
x_5	6		$2/7$		$-2/7$	1	
x_6	0		$3/7$		$-10/7$		1
L	24		2		-4		

Iteration 4

x_4	$7/2$	$7/12$	$-5/12$		1		
x_3	9	$3/2$	$-1/2$	1			
x_5	7	$1/6$	$1/6$			1	
x_6	5	$5/6$	$-1/6$				1
L	38	$7/3$	$1/3$				

After the first iteration we have obtained a system resolved with respect to the basic variables x_1, x_2, x_4, x_5 which is associated with the basic solution (6, 0, 0, 0, 6, 0) and the value $L_1 = 24$. The second and third iterations do not change the basic solution and the values $L_2 = L_3 = 24$, and only the fourth iteration results in the optimum solution (0, 0, 9, 7/2, 7/5) and $L_{max} = 38$. There was no cycling in this scheme of calculations, although it seems that we "marked time" in the course of three iterations since only basic and nonbasic variables changed. In the example we have considered, three equal minimum ratios, $b_2/a_{25} = b_3/a_{35} = b_4/a_{45} = 6$, turned out in the initial table. Therefore, using the rule of avoiding a possible cycling, we take the ratios between the elements of the column following the nonbasic column: $a_{26}/a_{25} = -1/5$, $a_{36}/a_{35} = -2$, $a_{46}/a_{45} = -2/3$.

The ratio $a_{36}/a_{35} = -2$ has proved to be the least. Consequently, the third row must be taken as the resolvent row, etc. (see tables).

1729. Maximize the linear function $L = 2x_1 + 4x_2$ subject to the constraints $-2x_1 + x_2 + x_3 = 6$, $-x_1 + (3/2)x_2 + x_4 = 9$, $-x_1 + 5x_2 + x_5 = 30$, $-x_1 + x_2 + x_6 = 12$, $x_1 \geqslant 0, x_2 \geqslant 0, x_3 \geqslant 0, x_4 \geqslant 0, x_5 \geqslant 0, x_6 \geqslant 0$.

11.4. Dual Problems

Every problem of linear programming can be associated with another problem having a certain connection with it. This other problem is called *dual* with respect to the first problem.

Thus if the initial problem (problem I) of linear programming consists in minimiz-

ing the linear function $L = c_1x_1 + c_2x_2 + \ldots + c_nx_n$, the constraints being given in the form of the inequalities

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geqslant b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \geqslant b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \geqslant b_m \end{cases}$$

provided x_k is nonnegative ($k = 1, 2, \ldots, n$), then it is associated with a dual problem (problem I′) in which it is required to **maximize** the linear function $T = b_1y_1 + b_2y_2 + \ldots + b_my_m$ subject to the constraints

$$\begin{cases} a_{11}y_1 + a_{21}x_2 + \ldots + a_{m1}y_m \leqslant c_1, \\ a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \leqslant c_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \leqslant c_n \end{cases}$$

provided y_i is nonnegative ($i = 1, 2, \ldots, m$).

Note that in problem I and the dual problem I′ the matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} \cdots a_{mn} \end{pmatrix} \quad \text{and} \quad A' = \begin{pmatrix} a_{11} & a_{21} \cdots a_{m1} \\ a_{12} & a_{22} \cdots a_{m2} \\ \ldots\ldots & \ldots\ldots \\ a_{1n} & a_{2n} \cdots a_{mn} \end{pmatrix},$$

compiled from the coefficients in the variables, are obtained from each other by means of transposition. The right-hand sides of the system of constraints of each problem contain coefficients of the linear function taken from the other problem. In the system of constraints of problem I (minimization) all the inequalities are of type "$\geqslant$", and in the system of constraints of problem I′ (maximization) all the inequalities are of type "$\leqslant$". The duality is mutual, that is, if problem I′ is written in the form analogous to problem I, then the initial problem I will turn out to be dual to it. Therefore, problems I and I′ are called *mutually dual* or *mutually conjugate*. It can be proved that $L_{\min} = T_{\max}$ and also that the necessary and sufficient condition for optimality of the solutions of any pair of dual problems is the equality $L(\bar{x}) = T(\bar{y})$, where $\bar{x}$ and $\bar{y}$ are feasible solutions of problems I and I′.

1730. Initial problem (I): find the nonnegative values (x_1, x_2) from the conditions $x_1 + 2x_2 \geqslant 4$, $x_1 - x_2 \geqslant -1$ and the minimization of the linear function $L = 3x_1 + 2x_2$.

The dual problem (I′): find the nonnegative values (y_1, y_2) from the conditions $y_1 + y_2 \leqslant 3$, $2y_1 - y_2 \leqslant 2$ and the maximization of the linear function $T = 4y_1 - y_2$.

Solution. We present here the geometrical solution of these problems. We construct the system of constraints of problems I and I′. The minimum of the linear function L is attained at the point $P(2/2; 5/3)$, i.e. $L_{\min} = 3 \cdot (2/3) + 2 \cdot (5/3) = 16/3$ and the maximum of the linear function T is attained at the point $P'(5/3; 4/3)$, i.e. $T_{\max} = 4 \cdot (5/3) - 4/3 = 16/3$ (Fig. 71).

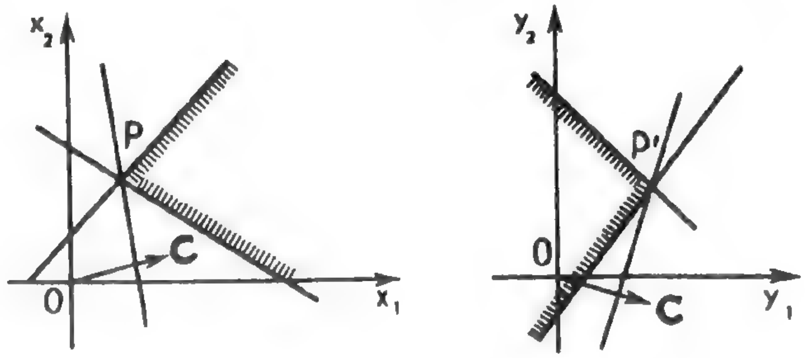

Fig. 71

1731. Initial problem (I): find the nonnegative values (x_1, x_2) minimizing the linear function $L = 3x_1 + 2x_2$ subject the constraints $7x_1 + 2x_2 \geqslant 14$, $4x_1 + 5x_2 \geqslant 20$. Derive the dual problem and solve it.

1732. Initial problem (I): find the nonnegative values (x_1, x_2) maximizing the linear function $L = 5x_1 + 4x_2$ subject to the constraints $4x_1 + 3x_2 \leqslant 24$, $3x_1 + 4x_2 \leqslant 24$. Derive the dual problem and solve it.

1733. Initial problem (I): find the nonnegative values (x_1, x_2) minimizing the linear function $L = 3x_1 + 3x_2$ subject to the constraints $5x_1 - 4x_2 \geqslant -2$, $x_1 + 2x_2 \geqslant 6$. Derive the dual problem and solve it.

11.5. Transportation Problem

The so-called *transportation problem* is one of the most typical problems of linear programming. It arises in planning the most rational ways of transporting loads. In some cases this means compiling a plan of transportation which will ensure the lowest cost of transportation, in other cases a gain in time is more important. The first problem has become known as the *cost criterion transportation problem* and the second, as the *time criterion transportation problem.*

The first problem is a special case of a linear programming problem and can be solved by the simplex method. But its peculiarities allow its simpler solution.

Assume that $a_1, a_2, \ldots, a_p$ units of homogeneous load are in p points of departure respectively. The load must be shipped to q consumers in the quantities of b_1, $b_2, \ldots, b_q$ units. Given are the costs c_{ik} of transportation of one unit from point i of departure to point k of destination. Let us denote $x_{ik} \geqslant 0$ $(i = 1, 2, \ldots, p; k = 1, 2, \ldots, q)$ the quantity of the units transported from the ith warehouse to the kth consumer; then the variables x_{ik} must comply with the following constraints: (1) $\sum\limits_{k=1}^{q} x_{ik} = a_i$ $(i = 1, 2, \ldots, p)$; (2) $\sum\limits_{i=1}^{p} x_{ik} = b_k$ $(k = 1, 2, \ldots, q)$; (3) $x_{ik} \geqslant 0$. The sum-total costs of transportation are $L = c_{11}x_{11} + c_{12}x_{12} + \ldots + c_{pq}x_{pq}$. Consequently, it is required to find pq variables x_{ik} satisfying the indicated conditions and minimizing the objective function L.

The solution of the problem should be divided into two stages:

(1) determination of the initial basic solution;

(2) construction of successive iterations, that is, approximation of the optimal solution.

11.5.1. Determination of the initial basic solution. Let us construct the initial basic solution by the so-called rule of "north-western corner". We tabulate the initial data of the problem as follows (see the table): we begin with the upper left corner and move either along the row to the right or along the column downwards. We put the smaller of the numbers a_1 and b_1 into the cell (1, 1), i.e. $x_{11} = \min\{a_1, b_1\}$.

If $a_1 > b_1$, then $x_{11} = b_1$ and the first column is "closed", that is, the requirements of the first consumer are completely satisfied. We move further along

a_i \ b_k	b_1	b_2	...	b_k	...	b_q
a_1	x_{11} c_{11}	x_{12} c_{12}	...	x_{1k} c_{1k}	...	x_{1q} c_{1q}
a_2	x_{21} c_{21}	x_{22} c_{22}	...	x_{2k} c_{2k}	...	x_{2q} c_{2q}
$\vdots$	....	....	...	....	...	....
a_i	x_{i1} c_{i1}	x_{i2} c_{i2}	...	x_{ik} c_{ik}	...	x_{iq} c_{iq}
$\vdots$	....	....	...	....	...	....
a_p	x_{p1} c_{p1}	x_{p2} c_{p2}	...	x_{pk} c_{pk}	...	x_{pq} c_{pq}

the first row putting the smaller of the numbers $a_1 - b_1$ and b_2 into the neighbouring cell (1, 2), i.e. $x_{12} = \min\{a_1 - b_1, b_2\}$.

Now if $b_2 > a_1$, then, by analogy, the first row is "closed" and we fill in the neighbouring cell (2, 1) where we put $x_{21} = \min\{a_2, b_1 - a_1\}$. The process is continued until at some stage the resources a_p are exhausted and the requirements b_q are satisfied.

1734. The amounts of 150 t and 90 t of fuel are at the points of departure A and B respectively. It is required to ship 60, 70 and 110 t of fuel to points *1*, *2* and *3* respectively. The costs of shipping one ton of fuel from point A to points *1*, *2*, *3*, respectively, are 6, 10 and 4 rubles and from point B, 12, 2 and 8 rubles. Compile the optimal plan of shipping the fuel so that the sum-total of transportation costs is the minimum.

Solution. Let us write down the initial data in Table 1. We begin with cell (1, 1): $x_{11} = \min\{150, 60\} = 60$, the first column is closed. We pass to cell (1, 2): $x_{12} = \min\{150 - 60, 70\} = 70$, the second column is closed; next we pass to cell

Table 1

a_i \ b_k	1: 60	2: 70	3: 110
A 150	60 [6]	70 [10]	20 [4]
B 90	[12]	[2]	90 [8]

Table 2

a_i \ b_k	1: 60	2: 70	3: 110	Remainder
A 150	60 [6]	[10]	90 [4]	60.0
B 90	[12]	70 [2]	20 [8]	20.0
Remainder	0	0	20.0	

(1, 3): $x_{13} = \min\{150 - 60 - 70, 110\} = 20$. Since there is a remainder in the third column, equal to 90, we pass to cell (2, 3) where we put $x_{23} = \min\{90, 90\} = 90$. Since the remainders in the row and the column are zero, the initial basic solution has been constructed. This plan envisages the costs $L = 6 \cdot 60 + 10 \cdot 70 + 4 \cdot 20 + 8 \cdot 90 = 1860$ rubles.

The rule of the "north-western corner" does not take into account the costs c_{ik} and therefore, the initial basic solution may be far from the optimal. Use can also be made of the method of "minimum element" which takes into account the quantity c_{ik}. In this case, the construction of the initial basic solution is begun from the cell with the least value of c_{ik}, in the given example from the cell (2, 2), where $c_{22} = 2$ (Table 2). This cell will contain $x_{22} = \min\{a_2, b_2\} = \min\{90, 70\} = 70$.

The remainders in the row and column are written in the respective cells of the row and the column of remainders. The column b_2 is closed. Now we pass to cell (1, 3) since after $c_{22} = 2$ the smallest is $c_{13} = 4$. The cell (1, 3) will contain $x_{13} = \min\{a_1 - b_1, b_3\} = \min\{150 - 60, 110\} = 90$. Next we pass to cell (1, 1): $x_{11} = \min\{a_1, b_1\} = \min\{150, 60\} = 60$, and, finally, we pass to cell (2, 3) in $x_{11} = \min\{a_1, b_1\} = \min\{150, 60\} = 60$, and, finally, we pass to cell (2, 3) in which we put $x_{23} = \min\{a_2 - b_2, b_3\} = \min\{90 - 70, 110\} = 20$.

Applying this rule, we have obtained another variant of the initial basic solution which envisages the costs $L = 6 \cdot 60 + 4 \cdot 90 + 2 \cdot 70 + 8 \cdot 20 = 1020$ rubles, that is, the sum of the costs is closer to the optimal plan.

11.5.2. Constructing successive iterations. Having obtained the initial basic solution, we now pass to constructing new basic solutions improving one another; for our purpose we apply the method of potentials.

Thus, after the initial basic solution has been constructed, all the variables turn out to be divided into two groups: x_{kl}, basic variables, and x_{pq}, nonbasic variables, and the linear functions representing the transportation costs are expressed in terms of the nonbasic variables as follows:

$$L = \sum_{pq} \gamma_{pq} x_{pq} + \gamma_0. \tag{1}$$

To find the coefficients γ_{pq} in the nonbasic variables, let us associate each point of departure A_i with some quantity u_i ($i = 1, 2, \ldots, m$), which will be called the *potential* of point A_i, and each point of destination B_j with the quantity v_j, the potential of point B_j. Let us relate these quantities by the equation $u_k + v_l = c_{kl}$, where c_{kl} is the cost of shipping one ton of cargo from point A_k to point B_l. It can be proved that the collection of equations $u_k + v_l = c_{kl}$ derived for all basic variables constitutes a consistent system of linear equations. In this case the value of one of the variables can be given arbitrarily, and the values of the other variables can be uniquely found from the system. For the nonbasic variables, the sum of the correspoding potentials will be denoted by c'_{pq}, i.e. $u_p + v_q = c'_{pq}$, and called an *indirect cost* (to distinguish it from the given cost c_{pq}). Then the coefficients in the nonbasic variables in relation (1) can be found with the aid of the equality $\gamma_{pq} = c_{pq} - c'_{pq}$.

If all the quantities γ_{pq} are nonnegative, then the initial solution is optimal. Now if there are negative quantities among them, we pass on to the next basis by increasing the term with the negative coefficient, leaving the rest of the variables equal to zero.

Let us make use of the general notions presented above and continue with problem 1734. We have obtained the initial basic solution (in accordance with the rule of the "minimum element"): $x_{11} = 60$, $x_{12} = 0$, $x_{13} = 90$, $x_{21} = 0$, $x_{22} = 70$, $x_{23} = 20$, $L = 1020$. To find the potentials, it is necessary to solve the system

$$u_1 = v_1 = c_{11} = 6, \quad u_1 + v_3 = c_{13} = 4, \; u_2 + v_2 = c_{22} = 2, \; u_2 + v_3 = c_{23} = 8.$$

We assign an arbitrary value to one of the unknowns, say, $u_1 = 1$. Then we have $v_1 = 5$, $v_3 = 3$, $u_2 = 5$, $v_2 = -3$. Next we calculate the indirect costs c'_{pq}:

$$c'_{12} = u_1 + v_2 = -2, \quad c'_{21} = u_2 + v_1 = 10.$$

Now we compute the differences $\gamma_{pq} = c_{pq} - c'_{pq}$:

$$\gamma_{12} = c_{12} - c'_{12} = 10 - (-2) = 12, \quad \gamma_{21} = c_{21} - c'_{21} = 12 - 10 = 2.$$

Consequently, the expression for L in terms of the nonbasic variables has the form $L = 1020 + 12x_{12} + 2x_{21}$. There are no negative coefficients in the variables on the right-hand side. This means that the initial basic solution is optimal.

Thus we see that the rule of the "minimum element" yields the optimal solution at once. Let us now solve the same problem under the condition that the initial solution has been obtained by the rule of the "north-western corner", i.e. $x_{11} = 60$, $x_{12} = 70$, $x_{13} = 20$, $x_{23} = 90$, $L = 1860$.

To find the potentials, we have to solve the system

$$u_1 + v_1 = c_{11} = 6, \quad u_1 + v_2 = c_{12} = 10, \quad u_1 + v_3 = c_{13} = 4, \quad u_2 + v_3 = c_{23} = 8.$$

Setting $u_1 = 1$, we get $v_1 = 5$, $v_2 = 9$, $v_3 = 3$, $u_2 = 5$.

We compute the indirect costs c'_{pq}:

$$c'_{21} = u_2 + v_1 = 10, \quad c'_{22} = u_2 + v_2 = 14.$$

Now we calculate the differences $\gamma_{pq} = c_{pq} - c'_{pq}$:

$$\gamma_{21} = c_{21} - c'_{21} = 12 - 10 = 2, \quad \gamma_{22} = c_{22} - c'_{22} = 2 - 14 = -12.$$

Consequently, the expression for L in the terms of the nonbasic variables has the form $L = 1860 + 2x_{21} - 12x_{22}$. The coefficient in x_{22} on the right-hand side is negative, hence we must try to decrease L by increasing x_{22} (retaining the zero value of x_{21}). We set $x_{22} = \lambda$. Since the sums of the values of the unknowns in the rows and columns must remain unchanged, it is necessary to perform the following balance recalculation:

60	$70 - \lambda$	$20 + \lambda$
	λ	$90 - \lambda$

The addition of λ to x_{22} is compensated for by the subtraction of λ from x_{12}, and this, in its turn, by the addition of λ to x_{13}, and so on, until we come back to x_{22}. Going through the cells along the dotted polygonal line, at one of whose vertices is the nonbasic variable x_{22} and at the other vertices, the basic variables (not necessarily all of them), we obtain the so-called *recalculation cycle* (the polygonal line is called a *cycle*), corresponding to the empty cell x_{22}. As is seen from the above table, to make the variables negative, we must increase λ to $\lambda = 70$. Then we get the second basic solution:

60	0	90
0	70	20

i.e. $x_{11} = 60$, $x_{12} = 0$, $x_{13} = 90$, $x_{21} = 0$, $x_{22} = 70$, $x_{23} = 20$.

For this solution, the value of the function L is $L = 1860 - 12 \cdot 70 = 1020$, that is, we have obtained the optimal solution (judging by the previous solution).

Thus, the rules of calculation by the method of potentials reduce to the following.

1. The potentials u_k and v_l are sought of all the points of departure A_k and the points of destination B_l.

2. Some nonbasic variable is chosen for which the sum of the potentials is strictly larger than the corresponding cost; this corresponds to the element with negative coefficient in the nonbasic variable on the right-hand side of the function L.

3. A recalculation cycle is found for the variable chosen in the previous item and this cycle is used to perform a shift which leads to a new feasible solution.

4. Operations 1-3 are performed repeatedly until the optimal basis is obtained, that is, nonnegative coefficients in the nonbasic variables on the rigth-hand side of the linear function L.

1735. Ninety tons of fuel are stored in each of two fuel depots A and B. Shipment of one ton of fuel from the warehouse A to the points *1*, *2*, *3* costs 1, 3, and 5 rubles respectively, and the shipment of one ton of fuel from B to the same points costs 2, 5 and 4 rubles respectively. Each receiving station must get the same amount of fuel. Compile a plan of shipping the fuel in accordance with which the transportation costs are at a minimum.

1736. Three railway stations A, B and C have reserved 60, 80 and 100 freight cars respectively. Compile the optimal plan of transferring these cars to four points of destination if point *1* requires 40 cars, point *2*, 60 cars, point *3*, 80 cars and point *4*, 60 cars. The costs of transferring one car from station A to the indicated points are equal, respectively, to 1, 2, 3, 4 rubles, from station B, 4, 3, 2, 0 rubles, and from station C, 0, 2, 2, 1 rubles.

1737. There are three shops A, B, C and four stores *1*, *2*, *3*, *4* in a factory. Shop A produces 30 000 articles, shop B, 40 000, and shop C, 20 000. The throughput of the stores for the same time period is characterized by the following data: store *1*, 20 000 articles, store 2, 30 000, store *3*, 30 000, store 4, 10 000. The costs of transporting 1000 articles from shop A to stores *1*, *2*, *3*, *4* are 2, 3, 2, 4 rubles respectively; from shop B, 3, 2, 5, 1 rubles, and from shop C, 4, 3, 2, 6 rubles. Compile a plan of transporting the articles which will envisage the minimum cost of transporting 90 000 articles.

1738. Three warehouses A, B, C contain high-quality grain in amounts of 10, 15, 25 tons respectively, which must be transported to four points: 5 tons to point *1*, 10 tons to point 2, 20 tons to point *3* and 15 tons to point *4*. The costs of transporting one ton of grain from warehouse A to the indicated points are equal, respectively, to 8, 3, 5, 2 rubles; from warehouse B, to 4, 1, 6, 7 rubles, and from warehouse C, to 1, 9, 4, 3 rubles. Compile the optimal plan of grain transportation to the four points, minimizing the transportation costs.

Answers

Chapter 1

4. (1) 8; (2) 3, **5.** (1) 1/2; (2) $-9/4$. **6.** $M(7)$. **7.** $C(1)$, $D(3)$. **8.** $C(-9)$, $D(-1)$. **16.** (1) 13; (2) 3. **19.** 5. **20.** $(-1; 8)$, $(1; 9)$, $(3; 10)$. **21.** $S = 0$, i.e. points A, B, C lie on one straight line. **22.** $D(17; 12)$. **23.** $C(-10; -7)$. **24.** $\sqrt{53}$, $\sqrt{82}$, $\sqrt{185}$. **25.** 24 sq. units. **29.** $A(4; \pi/6)$; $B(3; -\pi/2)$; $C(4\sqrt{2}; 3\pi/4)$; $D(2; -\pi/4)$; $E(2\sqrt{2}; 4\pi/3)$; $F(7; \pi)$. **30.** $A(0; 10)$; $B(-\sqrt{2}; -\sqrt{2})$; $C(0, 0)$; $D(\sqrt{2}/2; -\sqrt{2}/2)$; $E(-\sqrt{2}/2; -\sqrt{2}/2)$, $F(-\sqrt{2}/2; \sqrt{2}/2)$. **31.** $\sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos(\theta_1 - \theta_2)}$. **32.** 5. **33.** $M_1(\rho; -\theta)$. **34.** $M_1(\rho; \pi + \theta)$. **35.** (1) $(3; 7\pi/6)$, $(5; -\pi/3)$ and $(2; 5\pi/6)$; (2) $(3; -\pi/6)$, $(5; -2\pi/3)$ and $(2; \pi/6)$. **36.** $M_1(\rho; \pi - \theta)$. **44.** $y = 2x - 1.5$. **45.** Bisector of quadrants I and III. **46.** Bisector of quadrants II and IV. **47.** $x^2 + y^2 - 2x - 2y = 0$. **48.** $3x^2 + 2xy + 3y^2 - 4x - 4y = 0$. **49.** $\rho = a$. **50.** $\theta = \alpha$. **51.** $\rho = a\cos\theta$. **57.** Straight line $y = 2x$. **58.** $x^2/a^2 + y^2/b^2 = 1$ (the curve is an *ellipse*). **59.** $x^2/a^2 - y^2/b^2 = 1$ (the curve is a *hyperbola)*. **60.** The line segment (AB), where $A(1; 0)$, $B(0; 1)$. **61.** $x^{2/3} + y^{2/3} = a^{2/3}$. **62.** $x = a(t\sin t + \cos t)$, $y = a(\sin t - t\cos t)$ (the curve is an *involute of a circle*). **67.** (1) $x + 2y - 2\sqrt{5} = 0$. (2) $y = (-1/2)x + \sqrt{5}$. (3) $x/(2\sqrt{5}) + (y/\sqrt{5}) = 1$; (4) $(1/\sqrt{5})x + (2/\sqrt{5})y - 2 = 0$. **68.** 135°. **69.** 54 sq. units. **70.** No. **72.** $\sqrt{3}x + y - 1 = 0$. **73.** $x + y - 4 = 0$. **74.** $3x - 2y = 0$. **75.** $x + y - 7 = 0$. **76.** $x + 3 = 0$, $y + 4 = 0$. **77.** $x + y - 5 = 0$, $x + y + 5 = 0$. **99.** $\tan\alpha = 27/11$. **100.** $x - y = 0$, $5x + 3y - 26 = 0$, $3x + 5y - 26 = 0$. **101.** $14x + 14y - 45 = 0$, $2x - 2y + 35 = 0$. **102.** $3x - y + 14 = 0$, $x - 5y - 14 = 0$, $x + 2y = 0$. **103.** $x - 2 = 0$, $y - 7 = 0$. **104.** 4.4. **105.** 2.4. **106.** $m = 4$. **107.** $x - y = 0$, $x + 5y - 14 = 0$, $5x + y - 14 = 0$. **108.** $\pi/6$. **109.** $(0; 5)$ and $(4; 3)$. **110.** $(7/8; 0)$ and $(-27/8; 0)$. **111.** $13x + 6y - 82 = 0$, $3x + 4y - 23 = 0$, $S = 31.5$ sq. units. **112.** $3x - 2y = 0$, $5x + y + 6 = 0$. **113.** $5x + 4 = 0$. **114.** $5x + 8y + 11 = 0$. **115.** $5y + 2 = 0$. **116.** $17x + 11y = 0$. **117.** $x + y + 1 = 0$. **118.** $x = a$, $y = b$. **119.** $x = 1$; $y = x$. **120.** 30°. **121.** $\varphi = 53°8'$. **122.** $5x - 3y + 2 = 0$. **123.** $\sqrt{3}$ sq. units. **125.** $B(1; 3)$, $C(11; 6)$. **126.** (1) $x/4 + y/6 = 1$; (2) $x/4(\sqrt{2} - 1) + y/(-6)(\sqrt{2} + 1) = 1$; (3) $x/(-4)(\sqrt{2} + 1) + y/6(\sqrt{2} - 1) = 1$. **127.** $3x - 4y - 9 = 0$, $3x - 4y + 16 = 0$, $4x + 3y - 37 = 0$, or $4x + 3y + 13 = 0$. **134.** (1) $c = 4$, $b = -3$, $r = 5$; (2) $a = -5$; $b = 2$, $r = 0$; the equation specifies a point; (3) $a = 2$, $b = -7$, $r^2 = -1$; the equation is geometrically meaningless (an imaginary circle). **135.** $\tan\varphi = -2.4$. **136.** $(x + 1)^2 + (y - 1)^2 = 5$. **137.** $(x - 3)^2 + (y - 4)^2 = 25$. **138.** $x = 3.2$. **139.** $3x - 4y + 8 = 0$, $4x - 3y + 7 = 0$. **140.** $(x - 2)^2 + y^2 = 16$. **142.** $(4; 1.8)$; $(4; -1/8)$; $(-4; 1.8)$; $(-4; -1/8)$. **143.** b^2/a. **144.** $4x + 3y + 12 = 0$. **145.** $16x^2 + 25y^2 = 41$. **146.** Point M is outside the ellipse; point N is on the ellipse; point P is inside the ellipse. **147.** $e = \sin(\alpha/2)$. **148.** $M(-5; 7)$. **149.** $3x^2 + 3y^2 - 2xy - 2x - 2y - 1 = 0$. **150.** $x^2/3 + y^2/4 = 1$. **151.** The desired curve is an ellipse. If the coordinate axes are directed along the sides of the right angle (point A lies on the x-axis), the equation of the ellipse is $9x^2 + 36y^2 = 4a^2$. **155.** $x^2/9 - y^2/8 = 1$. **156.** $x^2/3 - y^2/5 = 1$. **157.** $(-4; -3)$. **158.** $x^2/64 + y^2/48 = 1$. **159.** $x^2 - y^2 = 8/225$. **160.** $e = 2/\sqrt{3}$. **161.** $(-8; 0)$. **162.** $x^2/4 -$

$-\ y^2/12 = 1$. **163.** 6 and 14. **166.** The right branch of the hyperbola $x^2 - y^2/3 = 1$. **169.** $y^2 = 4x$. **170.** $M_1(2;\ 4)$ and $M_2(2;\ -4)$. **171.** $y^2 = 4x$; $y^2 = -4x$. **172.** $y = \pm 2\sqrt{2}x$. **173.** $y^2 = \sqrt{2}x$. **174.** $M(0;\ 0)$, $M_1(18;\ -24)$. **175.** $y^2 = x$, $\tan\alpha = 8/15$. **179.** $(3;\ 2)$. **180.** $(8;\ -6)$. **183.** (1) $O_1(1;\ 2)$, $p = -1/4$; $x'^2 = -(1/2)y'$; (2) $O_1(1;\ 3)$, $p = -1/2$; $x'^2 = -y'$; (3) $O_1(1/16;\ 1/8)$, $p = -1/8$; $y'^2 = (-1/4)x'$; (4) $O_1(1;\ -2)$, $p = 1/2$; $y'^2 = x'$. **184.** (1) $x'y' = 1/8$; (2) $x'y' = 13/9$; (3) $x'y' = -6/5$; (4) $x'y' = 1/2$. **187.** The circle $(x - 1/2)^2 + (y - 1/3)^2 = 1$. **188.** The ellipse $x'^2/25 + y'^2/16 = 1$; the new origin is $O'(1;\ -1)$. **189.** The hyperbola $x'^2/4 - y'^2/9 = 1$; the new origin is $O'(2;\ 3)$. **190.** The point $O'(2;\ 1)$. **191.** The imaginary ellipse $x'^2/(-1) + y'^2/(-1/4) = 1$; $x' = x$, $y' = y + 1$. **192.** The hyperbola $y'^2 - x'^2 = 1$; the new origin is $O'(3;\ 0)$. **193.** The parabola $x'^2 = -y'$; the new origin is $O'(1;\ 5/2)$. **194.** The straight lines $x = 2$ and $x = 4$. **195.** Imaginary straight lines. **202.** The collection of two parallel straight lines $5x + y + 1 = 0$ and $5x + y - 1 = 0$. **203.** The collection of two merged straight lines $x + y + 1 = 0$. **204.** The collection of two intersecting straight lines $2x - 3y + 1 = 0$ and $4x - 3y - 1 = 0$. **205.** $x''^2/30 + y''^2/5 = 1$. **206.** $x''^2/9 - y''^2/36 = 1$. **207.** $y''^2 = -2x''$. **210.** $x = 1/2$, $y = 1/2$. **211.** The system is contradictory (it has no solutions). **212.** $x = a + b$, $y = a - b$. **213.** The system is indeterminate (it has an infinite number of solutions; x remains arbitrary, and $y = -(3/2)x + 1/12$. **214.** $x = y = z = t$. **215.** $x = \cos\alpha$, $y = \sin\alpha$. **216.** $x = 2t$, $y = t$, $z = 2t$. **222.** 0. **223.** 2. **224.** $2(ad - bc)$. **225.** $x = 1$, $y = 2$, $z = 3$. **226.** $x = 0$, $y = 0$, $z = -2$. **227.** $x = 0$, $y = 0$, $z = 0$. **228.** $x = t$, $y = 2t$, $z = -3t$. **229.** $x = 1$, $y = -1$, $z = 0$. **230.** $x = t$, $y = t$, $z = -t$.

Chapter 2

234. $C(5/3;\ 11/3;\ 13/3)$, $D(1/3;\ 13/3;\ 17/3)$. **236.** $M(3;\ 1;\ 3)$. **237.** In two. **238.** $M(0;\ 0;\ 17/8)$. **239.** $M(16;\ -5;\ 0)$. **245.** $\overline{AM} = (\mathbf{b} + \lambda\mathbf{c})/(1 + \lambda)$. **247.** $a_x = 0$, $a_y = 2$, $a_z = -2$. **248.** $m^2 + m + 1$. **250.** $a = 3/5$; $\cos\alpha = 1/3$, $\cos\beta = \cos\gamma = 2/3$. **251.** $|\overline{M_1M_2}| = 7$; $\cos\alpha = 2/7$, $\cos\beta = -6/7$, $\cos\gamma = 3/7$. **252.** $\mathbf{b} = -2\mathbf{j} + 5\mathbf{k}$ or $\mathbf{b} = -2\mathbf{j} - 5\mathbf{k}$. **253.** $M(-4;\ 4;\ 4\sqrt{2})$. **267.** -96. **268.** $\arccos(17/50)$. **269.** $m = 1$. **270.** 547. **271.** $A = \mathbf{Fs} = Fs\cos\varphi = 5\sqrt{3}$. **272.** $(\pm 1\sqrt{11})(\mathbf{i} - 3\mathbf{j} + \mathbf{k})$. **273.** $\mathbf{c} = \mathbf{i} + \mathbf{k}$ or $\mathbf{c} = (1/3)(-\mathbf{i} + 4\mathbf{j} - \mathbf{k})$. **274.** 20/3 and 20/7. **275.** $\mathbf{r}_D = 7\mathbf{i} + 7\mathbf{j} + 7\mathbf{k}$. **278.** No, since coplanar vectors cannot be pairwise perpendicular. **279.** $\mathbf{a} \times \mathbf{b} = -17\mathbf{i} + 7\mathbf{j} - \mathbf{k}$. **280.** $\sqrt{65}/2$ sq. units. **281.** 4. **283.** 20 cu. units. **284.** $4\sqrt{510}/17$.

Chapter 3

296. (1) $(x + y - z - 2)/\sqrt{3} = 0$; (2) $-3/(5\sqrt{2})x - (1/\sqrt{2})y + 4/(5\sqrt{2})z - 7/(5\sqrt{2}) = 0$. **297.** $d = 13/\sqrt{29}$; the origin and the point M_0 are on different sides of the plane. **298.** $d = 7\sqrt{5}/3$. **299.** (1) $x + y + z - 5 = 0$, (2) $2x + 2y + z - 6 = 0$. **300.** $7x - 11y - z - 15 = 0$. **301.** $M(5;\ 5;\ 5)$. **302.** $4x - 3y + 12z - 169 = 0$. **303.** $5y + 4z = 0$; $5x - 3z = 0$; $4x + 3y = 0$; **304.** $6x + 5y - 7z - 27 = 0$. **305.** $x/2 + y/2 + z/(\pm\sqrt{2}) = 1$. **306.** 60°. **307.** $x + 7y + 10z = 0$. **308.** $x - z = 0$. **309.** $x + y + z - 3 = 0$. **310.** $5x + 2y + 5z - 9 = 0$. **311.** $\sqrt{2}x + y + z - 5 = 0$. **312.** $4x + 3y - 2z - 1 = 0$. **313.** $(A_1D_2 - A_2D_1)x + (B_1D_2 - B_2D_1)y + (C_1D_2 - D_1C_2)z = 0$. **314.** $x - y + 2 = 0$. **315.** $\arcsin(5/6)$. **327.** $5y + 5z - 64 = 0$, $x = 0(yOz)$; $5x + 5z - 2 = 0$, $y = 0(xOz)$; $5x - 5y + 62 = 0$, $z = 0(xOy)$. **328.**

$(x+1)/5 = (y-3)/2 = z/1$. **329.** $\cos\alpha = 6/7, \cos\beta = 3/7, \cos\gamma = 2/7$. **330.** $(x-1)/\sqrt{2} = (y+2)/1 = (z-3)/(\pm 1)$. **331.** $(x-5)/1 = (y+1)/3 = (z+3)/(-11)$. **332.** $M(0; 7; -2)$. **333.** $(x-3)/(-1) = y/5 = (z+1)/2$; $x/2 = (y-7)/(-2) = (z+2)/3$. **334.** $x = -3t-1, y = 6t+1, z = t+2$. **335.** $5\sqrt{30}/6$. **336.** $(x-3)/3 = (y+1)/(-5) = (z-2)/(-2)$. **337.** $\cos\varphi = 20/21$. **338.** $x/0 = y/1 = z/2$. **339.** $(x-4)/2 = (y-1)/1 = (z+2)/(-2)$. **340.** $x/2 = (y-2)/(-1) = (z-1)/0$. **341.** $(x-1)/2 = (y-1)/(-3) = (z-1)/2$. **342.** $x/1 = (y-2)/(-1) = (z-1)/(-1)$. **343.** $x - 5y - 2z + 11 = 0$. **344.** $x/(-10) = (y-3.4)/13 = (z-5.2)/19$. **348.** (1) $C(-1; -2; 0), r = 5$; (2) $C(2; -3; -1), r = 4$; (3) $C(0; -1; 3/4), r = 3/4$; (4) $C(1; 0; 0;), r = 1$; (5) $C(0; 0; 2), r = 1$. **349.** (1) Inside the sphere; (2) outside the sphere; (3) on the sphere. **350.** $(x-2)^2 + (y-1)^2 + (z+2)^2 = 9$. **351.** $(x-1)^2 + (y-1)^2 = 16, z = 0$. **352.** $C(4; 4; -2); r = 8$. **356.** (1) Circular cylinder; (2) elliptic cylinder; (3) hyperbolic cylinder; (4) parabolic cylinder; (5) parabolic cylinder; (6) parabolic cylinder; (7) circular cylinder; (8) the z-axis, $x = 0, y = 0$; (9) bisector planes $x = z$ and $x = -z$; (10) the planes $y = 0$ and $y = x$. **357.** (1) $x^2 + z^2 = 9, y = 3$ (a circle); (2) $y^2 - x^2 = 1, z = 1$ (a hyperbola); (3) $z^2 - y^2 = 0, x = 0$ (two straight lines). **358.** (1) $y^2/b^2 + z^2/b^2 - x^2/a^2 = 0$; (2) $x^2/a^2 + z^2/a^2 - y^2/b^2 = 0$; (3) $x^2/a^2 + y^2/b^2 - z^2/c^2 = 0$. **361.** $4x^2 + 4y^2 - (z-2)^2 = 0$. **362.** $x^2 - y^2 = 1, z = 1$; $z + 1 = x^2, y = 1$; $y^2 = 1 - z, x = 1$; $y^2 - x^2 = 1, z = -1$. **363.** (1) Hyperbolic paraboloid; (2) cone with vertex at the origin. **364.** $3z = 2x^2 + y^2$. **365.** $x^2/9 + y^2/5 + z^2/1 = 1$. **366.** $x^2 + y^2 = 1, z = 1$ (a circle). **367.** (1) The axis of ordinates; (2) a cone with the y-axis and vertex at the origin; (3) a cone with the x-axis and vertex at the origin; (4) the origin; (5) a pair of planes intersecting along the z-axis. **374.** Two planes $x = y$ and $x = z$. **375.** A circular cylinder $(x-2)^2 + (z-2)^2 = 4$. **376.** The straight line $x = y = z$. **377.** The second-order cone $x^2 + (y-1)^2 - (z-1)^2 = 0$ with vertex $S(0; 1; 1)$. **378.** Point $(0; 1; -1)$. **379.** One-sheet hyperboloid with the canonic equation $x'^2 + y'^2/4 - z'^2/4 = 1$. **380.** Two-sheet hyperboloid with canonic equation $x'^2 + y'^2 - z'^2 = -1$. **381.** Paraboloid of revolution with canonic equation $x'^2 + y'^2 = 4z'$. **382.** Hyperbolic paraboloid with canonic equation $x'^2 - z'^2/9 = 2y'$.

Chapter 4

387. 900. **388.** 12. **389.** 21280. **390.** a^2b^2. **391.** $x = 1, y = 2, z = 1, t = -1$. **392.** $x = 1, y = 1, z = 0, t = -2$. **393.** $x = 1, y = 2, z = 3, u = 4, v = 5$.

412. $B = \begin{pmatrix} -4 & -8 & -4 \\ -3 & -1 & -5 \\ -7 & -6 & 1 \end{pmatrix}$. **413.** $\begin{pmatrix} 9 & 6 & 6 \\ 6 & 9 & 6 \\ 6 & 6 & 9 \end{pmatrix}$. **414.** $\begin{pmatrix} 1/10 & -1/5 & 7/10 \\ 0 & 1/10 & -1/5 \\ 0 & 0 & 1/10 \end{pmatrix}$.

416. $\lambda_1 = 2, \lambda_2 = 11$; $\mathbf{e}_1 = (4/\sqrt{41})\mathbf{i} - (5/\sqrt{41})\mathbf{j}$, $\mathbf{e}_2 = (1/\sqrt{2})\mathbf{i} + (1/\sqrt{2})\mathbf{j}$. **417.** $\lambda_1 = -2, \lambda_2 = 3, \lambda_3 = 6$; $\mathbf{r}_1 = \alpha(\mathbf{i} - \mathbf{k})$; $\mathbf{r}_2 = \beta(\mathbf{i} - \mathbf{j} + \mathbf{k})$, $\mathbf{r}_3 = \gamma(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$. **418.** $x''^2/16 + y''^2/4 = 1$. **419.** $x = 1, y = 2, z = 3$. **420.** $x''^2/25 - y''^2/9 = 1$. **421.** $y''^2 = 2/\sqrt{2}x''$. **422.** $x'^2 + y'^2/1 - z'^2/3 = 1$ (one-sheet hyperboloid). **423.** $2y''^2 + 3z''^2 = \sqrt{6}x''$ (elliptic paraboloid). **424.** $(t; 2t; 3t)$, where t is an arbitrary real number. **425.** $(2t; 2t, t)$, where t is an arbitrary real number. **426.** $(0; 0)$. **427.** The straight line $x'\cos\alpha - y'(1 + \sin\alpha) = 0$. **434.** $r(A) = 0$, if $\lambda = 0$; $r(A) = 2$, if $\lambda \neq 0$. **435.** $r(A) = 3$. **436.** $r(A) = 3$; base minors are

$\begin{vmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{vmatrix}$ and $\begin{vmatrix} 2 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 3 & 0 \end{vmatrix}$. **437.** $r(A) = 2$; base minors are $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$, $\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}$, $\begin{vmatrix} 1 & 3 \\ 3 & 6 \end{vmatrix}$, $\begin{vmatrix} 1 & 4 \\ 3 & 8 \end{vmatrix}$, $\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$, $\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}$, $\begin{vmatrix} 3 & 6 \\ 1 & 3 \end{vmatrix}$, and $\begin{vmatrix} 3 & 8 \\ 1 & 4 \end{vmatrix}$. **441.** The system is consistent, $r(A) = r(A_1) = 2$; $x_1 = 1, x_2 = 1/2$. **442.** $r(A) = 1, r(A_1) = 2$. The system is inconsistent. **443.** The system is consistent, $r(A) = r(A_1) = 2$. **446.** $x_1 = 1$, $x_2 = 5$, $x_3 = 2$. **447.** $x_1 = 1$, $x_2 = 2, x_3 = 3, x_4 = 4$. **448.** $x_1 = 5, x_2 = 4, x_3 = 3, x_4 = 1, x_5 = 2$. **449.** The system is inconsistent. **450.** $x = 1.96$, $y = 2.96$, $z = 5.04$. **451.** $x = 1.50$, $y = 1.16$, $z = 1.40$. **457.** $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4$. **458.** $x_1 = u, x_2 = u + 1, x_3 = u + 2, x_4 = u + 3$. **459.** The system is inconsistent. **460.** $r(A) = 3$.

Chapter 5

463. Yes. **464.** No, since the sum of two elements of a set is not the element of that set. **465.** No, since the sum of two polynomials of the second degree can be a polynomial of the first degree or a constant quantity. **466.** Yes. **467.** (1) Yes; (2) yes; (3) yes; (4) no. **468.** Yes. **469.** (1) Only if it is a null-vector; (2) no, since in this space, in addition to the vectors $\mathbf{x}$ and $\mathbf{y}$, there must also be other vectors of the form $\lambda\mathbf{x} + \mu\mathbf{y}$. **470.** No, since in the set of vectors obtained, there can be found vectors whose sum is equal to $\mathbf{x}$, say, vectors $(\mathbf{x} - \mathbf{y})/2$ and $(\mathbf{x} + \mathbf{y})/2$. **471.** It can be. For example, eliminating from the set of geometric vectors those which are not perpendicular to the Oz axis, we get the set of vectors $\lambda\mathbf{i} + \mu\mathbf{j}$ forming a linear space. **472.** (1) The collection of the systems of these numbers is a linear space; (2) the quantity of sugar sold exceeded the supply by 100 kg, the supply of tea exceeded by 5 kg the quantity sold, the whole supply of biscuits was sold, the quantity of cakes sold exceeded the supply by 200 kg, the supply of fruit exceeded by 3 kg the quantity sold. **473.** No, since $\lambda(\xi_1; \xi_2; \xi_3)$ does not belong to that set if λ is not an integer. **474.** No. **475.** No, since the oppositely-oriented vectors lie not in the first octant. **488.** The set of all polynomials not higher than of the nth degree. **501.** $\mathbf{x} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3 + 4\mathbf{e}_4$. **502.** $\mathbf{x} = \mathbf{e}_1' + \mathbf{e}_2' + \mathbf{e}_3' + \dots + \mathbf{e}_n'$. **504.** $\xi_1 = \varepsilon\xi_5', \xi_2 = \alpha\xi_1', \xi_3 = \beta\xi_2', \xi_4 = \gamma\xi_3', \xi_5 = \delta\xi_4'$. **505.** No, since the equality $\mathbf{e}_1' + \mathbf{e}_2' + \mathbf{e}_3' = 0$ is impossible by virtue of the linear independence of the basis vectors $\mathbf{e}_1'$, $\mathbf{e}_2'$ and $\mathbf{e}_3'$. **506.** It can, but only if this element is a null-vector. **508.** The intersection is the set of elements $\mathbf{x}_{12} = (0; 0; \xi_3; \xi_4)$, $\mathbf{y}_{12} = (0; 0; \eta_3; \eta_4)$, $\mathbf{z}_{12} = (0; 0; \zeta_3; \zeta_4)$, ... The sum coincides with the space R. **509.** $d(R_1) = 3$, $d(R_2) = 3$, $d(R_3) = 2$, $d(R_4) = 4$. **510.** No. **513.** R_3 is a set of constant quantities, R_4 is a set of polynomials of the form $c_0t^4 + c_1t^2 + c_2t + c_3$. **514.** R_3 is a set of all vectors parallel to the Ox axis, and $R_4 = R$. **516.** The set of all even functions forms a subspace, while the set of all odd functions does not, since the product of two odd functions is an even function. **517.** No, since any vector $\lambda\mathbf{a}$ does not belong to that set if λ is an irrational number. **522.** $k = 3$; $\mathbf{f}_1 = (-1; 0; 1; 0; 0)$, $\mathbf{f}_2 = (-1; 0; 0; 1; 0)$, $\mathbf{f}_3 = (0; -1/2; 0; 0; 1)$, $\mathbf{f} = (-c_1 - c_2; -0.5c_3; c_1; c_2; c_3)$. **526.** Yes. **527.** No, since the equality $|\mathbf{a} + \mathbf{b}|\cdot(\mathbf{a} + \mathbf{b}) = a\cdot\mathbf{a} + b\cdot\mathbf{b}$ is not satisfied if $ab \neq 0$. **528.** Only if $\mathbf{x}_0 = 0$. **529.** Only for $a = 0$. **530.** Yes.

533. $\begin{pmatrix} \alpha & 0 & 0 & \dots & 0 \\ 0 & \alpha & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \alpha \end{pmatrix}$. **536.** $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

538. $3A - 2B = E$. **544.** $A^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$. **545.** $A^{-1} = A$. **546.** $A^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. **547.** $B = 2E \cos\alpha$. **548.** The linear transformation A has no inverse since $|A| = 0$. **549.** $A^{-2} = (A^{-1})^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. **550.** For $\lambda = -2$. **552.** (1) If $\alpha \neq \beta$, then $\lambda_1 = \alpha$, $\mathbf{u} = c_1\mathbf{e}_1$, $\lambda_2 = \beta$, $\mathbf{v} = c_2\mathbf{e}_2$; (2) if $\alpha = \beta$, then $\lambda_1 = \lambda_2 = \alpha$, $\mathbf{u} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2$. **553.** $\lambda_1 = \lambda_2 = 2$, $\mathbf{u} = c_1(\mathbf{e}_1 + \mathbf{e}_2)$. **554.** (1) If $b \neq 0$, then the linear transformation does not have eigenvectors; (2) if $b = 0$, then $\lambda_1 = \lambda_2 = a$, $\mathbf{u} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2$. **556.** $\lambda = 2$, $\mathbf{u} = c_1(\mathbf{e}_1 - \mathbf{e}_3)$; $\lambda = 3$, $\mathbf{v} = c_2(\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3)$; $\lambda = 6$, $\mathbf{w} = c_3(\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3)$. **558.** $\lambda = -1$, $\mathbf{u} = c_1\mathbf{i} + c_2\mathbf{j}$. **560.** $\lambda = 1$, $\mathbf{u} = c_1(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)$; $\lambda = -1$, $\mathbf{v} = c_2(\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_4)$. **561.** $\lambda = \alpha + \beta + \gamma$, $\mathbf{u} = c(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$. **563.** (x, y) is the total cost of all the articles manufactured at the factory. **565.** Yes. **566.** No, since at $\lambda < 0$ conditions 2° and 3° are not satisfied. **567.** Yes. **569.** $\arccos(1/n)$. **573.** Yes. **576.** $|\mathbf{x}| = 5$. **577.** $\mathbf{x}/|\mathbf{x}| = (1/15)\mathbf{e}_1 + (2\sqrt{2}/15)\mathbf{e}_2 + (\sqrt{3}/5)\mathbf{e}_3 + (8/15)\mathbf{e}_4 + (\sqrt{5}/3)\mathbf{e}_5$. **579.** $\mathbf{x}$ is a unit vector. **580.** $\varphi = \pi/3$. **581.** $\pm 0.5(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)$. **582.** $\lambda = \pm 1$. **587.** Yes. **588.** Yes. **589.** At $\lambda = \pm 1$. **590.** Yes, since the vectors $A\mathbf{e}_1$, $A\mathbf{e}_2$ and $A\mathbf{e}_3$ form an orthonormal basis. **591.** Yes.

Chapter 6

602. $n = 4$. **603.** $\delta = 0.16\%$. **604.** $\delta = 0.0005\%$. **605.** $\delta = 0.022\%$; $n = 4$; $S = 8765 \pm 0.1$ (sq. m). **613.** (1) $[-2, 0) \cup (0, 2]$; (2) $[0, 4]$; (3) $(-\infty, 0) \cup (0, +\infty)$; (4) $x \neq \pi(2n + 1)/4$; $n \in Z$; (5) $(-\infty, -2] \cup [2, +\infty)$; (6) $(1/3, +\infty)$; (7) $[0, 2]$. **614.** (1) $[1, +\infty)$; (2) $(-\infty, 0) \cup (0, +\infty)$; (3) $[-4, 4]$; (4) $(-\infty, 3)$; (5) $[-2, 4]$; (6) $(0, 1]$. **615.** (1) Odd; (2) even; (3) neither even nor odd; (4) even; (5) neither even nor odd; (6) even; (7) odd. **616.** (1) $2\pi/5$; (2) 6π; (3) π; (4) π. **653.** 1/2. **654.** -1. **655.** 1/6. **656.** -2. **657.** $-\sin a$. **658.** m/n. **659.** $\sec^2 x_0$. **660.** $-\sqrt{2}/4$. **661.** 1/2. **662.** ∞. **663.** 2. **664.** 3/4. **665.** $-1/4$. **666.** 1/2. **667.** 3. **668.** $\sqrt{7}/4$. **669.** 25/9. **670.** 1/2. **671.** m. **672.** 1, if $x \to +\infty$; -1, if $x \to -\infty$. **673.** $(a - c)/2$. **674.** 0. **675.** 0. **676.** $\ln 5$. **677.** $\ln(8/7)$; $\ln(6/5)$. **678.** 2. **679.** $\ln 5$. **680.** 1/4. **681.** 1, if $x \to +0$; -1 if $x \to -0$. **682.** $+\infty$. **683.** 2. **684.** 0. **685.** Does not exist. **686.** 5/4. **687.** $\ln a$. **688.** e. **689.** e^3. **690.** 1/6. **691.** $\ln(5/4)$. **692.** 1. **693.** -3. **694.** 0. **695.** 1/2. **696.** e^{10}. **697.** $\sqrt{e}$. **698.** e^{a-b}. **699.** $\sqrt{e}$. **704.** $y \sim x$. **705.** 2. **706.** 1/2. **707.** $\alpha = o(\beta)$. **708.** $\alpha \sim \beta$. **709.** $\alpha \sim \beta$. **710.** 1/3. **711.** 9/4. **712.** $-1/2$. **713.** $-1/2$. **714.** $-1/2$. **715.** 1. **716.** 9/25. **717.** $(\ln 5 \cdot \ln 4)/(\ln 3 \cdot \ln 6)$. **718.** 1.6. **722.** $x = 2$ is a point of jump discontinuity. **723.** $x = 1$, $x = 5$ are points of discontinuity of the second kind. **724.** Discontinuity of the second kind. **725.** $x = 0$ is a point of removable discontinuity. **726.** $x = 3$ is a point of jump discontinuity; $x = 5$ is a point of discontinuity of the second kind; $x = 0$ is a point of removable discontinuity; $x = \pi/2 + \pi n\, (n \in Z)$ are points of discontinuity of the second kind. **727.** $x = 1$, $x = 2$ are

points of removable discontinuity: **728.** $x = -2$, $x = -3$ are points of discontinuity of the second kind; $x = -1$ is a point of removable discontinuity. **729.** The function is continuous on an infinite interval $(-\infty, +\infty)$. **730.** (1) The function is continuous; (2) possesses one point of discontinuity of the second kind; (3) possesses two points of discontinuity of the second kind. **731.** (1) The function is continuous; (2) possesses two points of discontinuity of the second kind; (3) possesses four points of discontinuity of the second kind.

Chapter 7

735. $y' = -2/x^3$. **736.** $y' = 2/(3\sqrt[3]{x})$. **737.** $y' = 5\cos x - 3\sin x$. **738.** $y' = 5\tan^2 x$. **739.** $y' = -e^x/(e^x+1)^2$. **740.** $y' = 2^{x^2}\cdot 2x\ln 2$. **767.** $y' = -21/x^4$. **768.** $y' = \sqrt[3]{x}$. **769.** $y' = x^2\sqrt{x}(1-x^2)^2$. **770.** $y' = -x^2e^{-x}$. **771.** $y' = 9x^2\ln x$. **772.** $y' = (8/9)^x\ln(8/9)$. **773.** $y' = x^2\cos x$. **774.** $y' = 6(x+1)/(2x^2+3x)$. **775.** $y' = -3x/\sqrt{1-3x^2}$. **776.** $y' = \arccos(x/2)$. **777.** $y' = 1/(2\sqrt{x})\arcsin\sqrt{x}$. **778.** $y' = -\cos x$. **779.** $y' = -\cos^2(x/3)\sin(x/3)$. **780.** $y' = \operatorname{cosec}(2x+1)/2$. **781.** $y' = 1/\cos x$. **782.** $y' = 2\sec^6 2x$. **783.** $y' = 1/(2\sqrt{x})\sin^2\sqrt{x}\cos^5\sqrt{x}$. **784.** $y' = 6x\sqrt{9x^4+1}$. **785.** $y' = \sqrt{a^2-x^2}$. **786.** $y' = -2\sec^2 x/\sqrt{\tan x}(4\tan x+1)$. **787.** $y' = \cot^3(x/2)$. **788.** $y' = 1/(x\sqrt{4x^2-1})$. **789.** $y' = 1/\sqrt{a^2-x^2}$. **790.** $y' = 1/(2\sqrt{1-x^2})$. **791.** $y' = 6x^2/(1+x^6)$. **792.** $y' = 6\operatorname{sign} x/(x^2+9)$. **793.** $y' = -2e^{-x}\sin^2 e^{-x}$. **794.** $y' - 1/(2\sqrt{1-x^2})$. **795.** $y' = -6/[(x-1)(x-2)(x-3)(x-4)]$. **796.** $y' = 3e^{\sin^2 3x}\sin 6x\sin^2 3x$. **797.** $y' = -24\ln\sin x\cdot\cot x/(4\ln^4\sin x - 9)$. **798.** $y' = \sec x$. **799.** $y' = \operatorname{cosec} x$. **800.** $y' = e^{\sqrt{2x}}$. **801.** $y' = 10/[x(x^5+2)]$. **802.** $y' = 2\sin x/(1+\cos x)^2$. **803.** $y' = \cos x/(1+\sin^2 x)$. **804.** $y' = \operatorname{cosec}^2(x/2)\cot(x/2)$. **805.** $y' = (2/x)\cos^2(\ln x)$. **806.** $y' = 5x^4\ln(x^5+3)$. **807.** $y' = -x/(|x|\sqrt{5-x^2})$. **808.** $y' = \sqrt{x^2+2\alpha x+\beta}$. **809.** $y' = 0$. **810.** $y' = (mx+n)/\sqrt{x^2+2\alpha x+\beta}$. **811.** $y' = 1/(1-mx^2)^{3/2}$. **812.** $y' = 4x\cos^2 x$. **813.** $y' = -2\operatorname{cosec}^3 x$.

814. $y' = \dfrac{\cos x\ln\sin x}{(1+\ln\sin x)^2}$. **815.** $y' = 9x\sin^2 x\cos x$. **816.** $y' = 2/(x\sqrt{1+x^2})$. **817.** $y' = 2e^2\sin^2 e^x$.

818. $y' = 2/(x^2+2x+2)^2$. **819.** $y' = \ln^3 x$. **820.** $y' = (\ln\sin\sqrt{x})/(2\sqrt{x}\cos^2\sqrt{x})$. **821.** $y' = 2(1+\ln x)/(x^x+x^{-x})$. **822.** $y' = 1/\sin^5 x$. **823.** $y' = \cos^4 x/\sin x$. **824.** $y' = 1/(3\sin x+4\cos x+5)$. **825.** $y' = \cos x\tan^3\sin x$. **826.** $y' = 1/[x^2(x-1)]$. **827.** $y' = 1/[x(x+1)(x+2)]$. **828.** $y' = 4x\tan^2 2x$. **829.** $y' = -2e^x/\sqrt{1-e^{2x}}$. **830.** $y' = (\ln\ln\ln x)/(x\ln x)$. **831.** $y = 2e^{2x}(1-2x)/(x+e^{2x})^2$. **832.** $y = 2(\ln x+1)/(x^2\ln^2 x-1)$. **833.** $y' = 3/(1+x^2)$. **834.** $y'(e^{2\sin x}\cos x) : \sin(e^{2\sin x}/2)$. **835.** $y' = a\sin 2x$. **836.** $y' = 3\sec^2 x\sec^4\tan x$. **837.** $y' = (-1/x^2)\arctan x$. **838.** $y' = (-5/x^6)\ln x$. **839.** $y' = (1/\sqrt{2x+1})\ln(2x+1)$. **840.** $y' = \sec x\tan x\ln\sec x$. **841.** $y' = -2e^{3x}/\sqrt{1-e^{2x}}$. **842.** $y = 3\cdot 2^{\cos^3 x-3\cos x}\cdot\sin^3 x\cdot\ln 2$. **843.** $y = (e^x\cdot 2^{5x}/3^{4x})\cdot\ln(32e/81)$. **844.** $y' = 0$. **845.** $y' = x\cos 2x$. **846.** $y' = 2e^{x^2}(x^4+x^2+1)/(x^2+1)^2)$. **847.** $y' = x\cos x/\sin^2 x$. **848.** $y' = (1/\sqrt{x})\tan^2\sqrt{x}$. **849.** $y = x^2/(x^4-a^4)$. **850.** $y' = -4x/\sqrt{x^4+1}$. **851.** $y = e^{0.5\tan^2 x}\times\sin x\tan^2 x$. **852.** $y' = 8x^3/(1+x^8)$. **853.** $y' = xe^{x^2}(2x^2\ln x+2\ln x+1)$. **854.** $y' = 0.5\ln 2\sqrt{2^x}/(1-2^x)$. **855.** $y' = -\ln 2/(2x\ln^2 x)$. **856.** $y' = (mx+n)/\sqrt{-x^2+2\alpha x+\beta}$.

857. $y' = (2/\ln 2)\cot x$. **858.** $y' = 1/(\sqrt{x^2+9}\ln a)$. **859.** $y' = x^{\arcsin x}\left(\dfrac{\ln x}{\sqrt{1-x^2}} + \dfrac{\arcsin x}{x}\right)$.

860. $y' = 8(x-1)^3/(x+1)^5$. **861.** $y' = \dfrac{2^x(x+1)^3}{(x-1)^2\sqrt{2x+1}}\left(\ln 2 + \dfrac{3}{x+1} - \dfrac{2}{x-1} - \dfrac{1}{2x+1}\right)$.

862. $y' = -1(|x|\sqrt{1+x^2})$. **863.** $y' = x^m\cos(n\ln x)$. **864.** $y' = (x\tan x + \ln\cos x)\sec^2(x\tan x + \ln\cos x)\,x\sec^2 x$. **865.** $y' = -x\sin x\cdot\ln(x\cos x - \sin x)$. **866.** $y' = 3\cos^3(xe^x - e^x)\times xe^x$. **867.** $y' = -2(1-2x^2)/(|1-2x^2|\sqrt{1-x^2})$. **868.** $y' = \begin{cases} 1, & \text{if } x>0; \\ -1, & \text{if } x<0. \end{cases}$ **869.** $y' = \begin{cases} f'(x), & \text{if } f(x)>0; \\ -f'(x), & \text{if } f(x)<0. \end{cases}$ **870.** $y' = \begin{cases} 3, & \text{if } x>5/3; \\ -3, & \text{if } x<5/3. \end{cases}$ **871.** $y' = \begin{cases} e^x, & \text{if } x>0; \\ -e^{-x}, & \text{if } x<0. \end{cases}$

872. $y' = \begin{cases} -2, & \text{if } x<0; \\ 0, & \text{if } 0<x<2; \\ 2, & \text{if } x>2 \end{cases}$ **873.** $y' = 2xe^x\sin x$. **874.** $y' = (x\cos x)/\sqrt{(x\sin x + \cos x)^2 + 1}$.

875. $y' = x^{x+1}\ln x(\ln x - 1)/e^x$. **876.** $y' = (\cot x\cdot\ln\cos x + \tan x\cdot\ln\sin x)/\ln^2\cos x$. **877.** $y' = nx^{n-1}/(2\sqrt{x^{2n}+1})$. **878.** $y' = -(\log_x e)^2/x$. **879.** $y' = 0$. **880.** $y' = 1/2$. **881.** $y' = 0$. **882.** $y' = x^x(1+\ln x)$. **883.** $y' = x^{-x}\cdot 2^x\cdot x^2(\ln 2 + 2x^{-1} - 1 - \ln x)$. **884.** $y' = 2x^{\ln x - 1}\ln x$. **885.** $y' = \dfrac{x^2\sqrt{x+1}}{(x-1)^3\sqrt[5]{5x-1}}\left[\dfrac{2}{x} + \dfrac{1}{2(x+1)} - \dfrac{3}{x-1} - \dfrac{1}{5x-1}\right]$. **889.** $(\operatorname{arcsec} x)' = 1/(x\sqrt{x^2-1})$, $(\operatorname{arccosec} x)' = -1/(x\sqrt{x^2-1})$. **890.** $4\operatorname{cosec}^2 x$. **894.** $y' = (x^2-y)/(x-y^2)$. **895.** $y' = -(Ax+By+D)/(Bx+Cy+E)$. **896.** $y' = x/(3y)$. **897.** $y' = y(y - x\ln y)/x(x - y\ln x)$. **898.** $y' = -(y\cos x + \sin y)/(x\cos y + \sin x)$. **899.** $y' = -(e^x - y\cdot 2^{xy}\times\ln 2)/(e^y - x\cdot 2^{xy}\cdot\ln 2)$. **900.** $y' = 2x$. **901.** $y' = y/x$. **902.** $y' = -y/(2x\ln x)$. **903.** $y' = -(2x\sin y - y^3\sin x - 2)/(x^2\cos y + 3y^2\cos x - 3)$. **905.** $-\cot t$. **906.** e^{2t}. **907.** $2\sqrt{\alpha}\;e^{-\sqrt{\alpha}}$. **908.** $\coth t$. **910.** $\alpha = \pi/4$; $y = x + 1$. **913.** $y - y_0 = (-y_0/p)(x - x_0)$. **914.** $x - y + 1 = 0$. **915.** $x + y - 1 = 0$; $x - y = 0$. **916.** $x - y + 2 - \pi/2 = 0$; $x + y - \pi/2 = 0$. **917.** $3x - y - 4 = 0$; $x + 3y - 28 = 0$. **919.** $\pi/4$. **920.** $3x - 8y + 10 - 6\ln 2 = 0$; $32x + 12y - 15 - 64\ln 2 = 0$. **921.** $x\cosh t_0 - y\sinh t_0 - 1 = 0$. **922.** $\tan\varphi = 2/3$. **923.** $\pi/4$. **924.** $\pi/4$; $3\pi/4$. **925.** $\tan\varphi_1 = 3$, $\tan\varphi_2 = -3$. **926.** $\pi/2$. **929.** 1.76 m/s. **930.** $x'_t = \sqrt[3]{a}$. **936.** $y'' = -44/(x+5)^3$. **937.** $y'' = \ln x$. **938.** $y'' = 2\sqrt{1-x^2}$. **939.** $y'' = x\sin 3x$. **940.** $y'' = 1/\sqrt{x^2+a^2}$. **941.** $\dfrac{d^2y}{dx^2} = -\dfrac{1}{4a}\operatorname{cosec}^4\dfrac{t}{2}$. **942.** $\dfrac{d^2y}{dx^2} = -4\sqrt{t-t^2}$. **945.** $-1/32$ m/s^2. **946.** $y''' = 1/(x+1)^4$. **947.** $y''' = (2\ln x - 3)/x^3$. **948.** $y''' = 105\sqrt{2x+3}$. **949.** $y''' = 4\sinh 2x$. **950.** $y^{(n)} = \dfrac{3\cdot 5\cdot 7\ldots(2n+1)}{2^n}\sqrt{x}$. **951.** $y^{(n)} = \dfrac{n!(-2)^n}{(2x+1)^{n+1}}$. **952.** $y^{(n)} = -1.5\cdot 2^n\cos(2x + \pi n/2)$. **953.** $y^{(n)} = [2^x + (-1)^n\times 2^{-x}]\ln^n 2$. **954.** $y^{(n)} = \dfrac{n!(ad-bc)(-c)^{n-1}}{(cx+d)^{n+1}}$. **955.** $y^{(n)} = k^n e^{kx}$. **956.** $y^{(n)} = \cos(x + \pi n/2)$.

957. $\dfrac{d^ny}{dx^n} = (-1)^n \cdot \dfrac{1}{t}$. **958.** $y''' = y^{IV} = \ldots = 0$. **968.** $dy = \sqrt{49 - x^2}dx$. **969.** $dy = \dfrac{dx}{x^2 - 36}$. **970.** $dy = \tanh(x/2)dx$. **971.** $dy = \dfrac{2e^{2x}dx}{1 + e^{4x}}$. **972.** $dy = \ln x\, dx$, $d^2y = \dfrac{(dx)^2}{x}$, $d^3y = -\dfrac{(dx)^3}{x^2}$.

973. $d^2y = \dfrac{-x(dx)^2}{(x^2 + 4)^{3/2}}$. **974.** $\Delta y = -\dfrac{\Delta x}{x(x + \Delta x)}$, $dy = -\dfrac{\Delta x}{x^2}$. **975.** $\Delta y = 0.0401$; $dy = 0.04$.

976. 0.811. **977.** 34.04 m^3. **978.** 1.035. **979.** 0.078. **980.** $\pi/4 + 1/13$. **981.** 1.9938. **993.** $\xi = 5/2$. **994.** $\dfrac{1}{x_0} - \dfrac{x - x_0}{x_0^2} + \dfrac{(x - x_0)^2}{x_0^3} - \dfrac{(x - x_0)^3}{x_0^4} + R_3$, where $R_3 = \dfrac{(x - x_0)^4}{\xi^5}(x_0 \lesseqgtr \xi \lesseqgtr x)$. **995.** $M(\sqrt{3}; 0)$. **996.** 0.754. **997.** 4.946. **998.** 1.395. **999.** 2.002. **1000.** 0.587. **1010.** 3/5. **1011.** 2. **1012.** 2/3. **1013.** 1/3. **1014.** 0.18. **1015.** 18. **1016.** 1. **1017.** 0. **1018.** 1/2. **1019.** ∞. **1020.** 0. **1021.** $1/\pi$. **1022.** 1. **1023.** 0. **1024.** $-1/2$. **1025.** $(p - q)/2$. **1026.** 2/3. **1027.** 1. **1028.** e^{-6}. **1029.** 2. **1030.** $e^{1/3}$. **1041.** Increases on $(-\infty, -1)$ and on $(1, +\infty)$, decreases on $(-1, 1)$. **1042.** Decreases on $(-\infty, -1)$, increases on $(-1, +\infty)$. **1043.** Increases on $(-\infty, 1)$, decreases on $(1, +\infty)$. **1044.** Decreases on $(-\infty, -1)$ and on $(1, +\infty)$, increases on $(-1, 1)$. **1045.** $y_{min} = y(0) = 0$, $y_{max} = y(2\sqrt[3]{2/49}) = (12/49)\sqrt[3]{4/7}$. **1046.** $y_{max} = y(11/4) = 13/4$. **1047.** $y_{min} = y(0) = 0$. **1048.** $y_{min} = y(0) = 1$. **1049.** $y_{min} = y(e) = e$. **1050.** $y_{max} = y(1) = 1/\sqrt{e}$, $y_{min} = y(-1) = -1/\sqrt{e}$. **1051.** $y_{min} = y(1) = 0$. **1052.** $y_{min} = y(3) = 0$, $y_{max} = y(2) = 3$. **1053.** $y_{min} = y(1) = -1$. **1054.** $y_{max} = (\pi - 12 + 6\sqrt{3})/12$, $y_{min} = (5\pi - 12 - 6\sqrt{3})/12$. **1055.** $y_{max} = e^{3/2}$, $y_{min} = e^{-3/2}$. **1056.** $y_{least} = 2$, $y_{gr} = 66$. **1057.** (0; 4) and (0; -4). **1058.** 25 km; 8 h 15 min. **1059.** $5\sqrt{2}$, $3\sqrt{2}$. **1060.** $l/4$, $l/4$. **1061.** $V = 2\pi l^3\sqrt{3}/27$. **1062.** $V = (S/3)\sqrt{S/(6\pi)}$. **1063.** At the distance of 9 km from A. **1064.** 125 m. **1065.** $a/\sqrt{2}$. **1069.** Convex on $(-\infty, -2)$, concave on $(-2, +\infty)$ **1070.** (4; 20). **1071.** (1; 0). **1072.** There are no points of inflection. **1077.** $x = 0$; $y = 2x$. **1078.** $x = 0$; $y = -3x$. **1079.** $y = x - 6$. **1080.** $y = x/2 + \pi$ and $y = x/2$. **1081.** $y = \pi x/2 + 1$ and $y = -\pi x/2 + 1$. **1084.** $D(y) = (-\infty, +\infty)$; the function is even and periodic with period π. It increases on $(\pi k, \pi/2 + \pi k)$, decreases on $(\pi/2 + \pi k, \pi + \pi k)$; $y_{min} = y(\pi k) = 0$, $y_{max} = y(\pi/2 + \pi k) = 1$, $k \in Z$. The curve is concave on $(-\pi/4 + \pi k, \pi/4 + \pi k)$ and convex on $(\pi/4 + \pi k, 3\pi/4 + \pi k)$; the points of inflection are $(-\pi/4 + \pi k; 1/2)$ and $(\pi/4 + \pi k; 1/2)$, $k \in Z$. **1085.** $D(y) = (-\infty, +\infty)$, the function is odd. It decreases on $(-\infty, -1)$ and on $(1, +\infty)$, increases on $(-1, 1)$; $y_{min} = y(-1) = -2$, $y_{max} = y(1) = 2$. The curve is concave on $(-\infty, 0)$ and convex on $(0, +\infty)$; the point of inflection is (0; 0). **1086.** $D(y) = (1, +\infty)$; the asymptotes are $x = 1$, $y = 0$. It decreases throughout the domain of definition. The curve is concave everywhere. There are no extrema or points of inflection. **1087.** $D(y) = (-\infty, 0) \cup (1, +\infty)$; the asymptotes are $x = 0$, $x = 1$, $y = 0$. Increases on $(-\infty, 0)$ decreases on $(1, +\infty)$. The curve is concave everywhere. There are no extrema or points of inflection. **1088.** $D(y) = (-\infty, -2) \cup (-2, 2) \cup (2, +\infty)$; the function is odd; the asymptotes are $x = -2$, $x = 2$, $y = x$. Increases on $(-\infty, -2\sqrt{3})$ and on $(2\sqrt{3}, +\infty)$, decreases on $(-2\sqrt{3}, -2)$, $(-2, 2)$ and on $(2, 2\sqrt{3})$; $y_{min} = y(2\sqrt{3}) = 3\sqrt{3}$, $y_{max} = y(-2\sqrt{3}) = -3\sqrt{3}$. The curve is convex on $(-\infty, -2)$ and on $(0, 2)$, concave on $(-2, 0)$ and on $(2, +\infty)$; the inflection point is (0; 0). **1089.** $D(y) = (0, +\infty)$; the asymptote is $y = 0$. Increases on $(0, e^2)$, decreases on $(e^2, +\infty)$; $y_{max} = y(e^2) = 2/e$. The curve is convex on $(0, e^{8/3})$ and concave on $[e^{8/3}, +\infty)$; the point of inflection is $(e^{8/3};$

$8e^{-4/3}/3)$. **1090.** $D(y) = (-\infty, +\infty)$. Decreases on $(-\infty, 1/4)$, increases on $(1/4, +\infty)$; $y_{min} = y(1/4) = -27/16$. The curve is concave on $(-\infty, 1/2)$ and on $(1, +\infty)$, convex on $(1/2, 1)$; the inflection points are $(1/2; -1)$ and $(1; 0)$. **1091.** $D(y) = [0, +\infty)$. Decreases on $(0, 1/3)$, increases on $(1/3, +\infty)$; $y_{min} = y(1/3) = -2/(3\sqrt{3})$. The curve is concave everywhere. **1092.** $D(y) = (-\infty, +\infty)$; the asymptote is $y = x$. Decreases on $(-\infty, 0)$, increases on $(0, +\infty)$; $y_{min} = y(0) = 1$. The curve is concave everywhere. **1093.** $D(y) = (-\infty, +\infty)$; the function is odd; the asymptotes are $y = -1$, $y = 1$. Increases on $(-\infty, +\infty)$. The curve is concave on $(-\infty, 0)$ and convex on $(0, +\infty)$; the inflection point is $(0; 0)$. **1094.** $D(y) = (-\infty, +\infty)$, the asymptote is $y = 0$. Increases on $(-\infty, 1)$, decreases on $(1, +\infty)$; $y_{max} = y(1) = e$. The curve is concave on $(-\infty, 1 - \sqrt{2}/2)$ and on $(1 + \sqrt{2}/2, +\infty)$, convex on $(1 - \sqrt{2}/2, 1 + \sqrt{2}/2)$; inflection points are $(1 + \sqrt{2}/2; \sqrt{e})$ and $(1 - \sqrt{2}/2; \sqrt{e})$. **1095.** $D(y) = (-\infty, 2) \cup (2, +\infty)$; the asymptotes are $x = 2$ and $y = x + 4$. Increases on $(-\infty, 2)$ and on $(6, +\infty)$ decreases on $(2, 6)$; $y_{min} = y(6) = 27/2$. The curve is convex on $(-\infty, 0)$, concave on $(0, 2)$ and on $(2, +\infty)$ the point of inflection is $(0; 0)$. **1100.** $R = 25/3$. **1101.** $R = (4a/3)\cos(\theta/2)$. **1102.** $k = 1/(e\sqrt{2})$. **1103.** $(2; 2)$. **1104.** A circle $\xi^2 + \eta^2 = 1$. **1107.** The first. **1108.** The first. **1109.** The third. **1110.** The first. **1111.** The third. **1112.** The second. **1114.** A circle $x^2 + z^2 = 1$, $y = 1$. **1115.** A straight line passing through the origin and forming equal angles with the coordinate axes. **1116.** A straight line which is parallel to the Oz axis and passes through the point $(1; 1; 0)$.

1117. An equilateral hyperbola lying in the plane xOz. **1120.** $(\mathbf{i} + \mathbf{k})\sinh 2t + \mathbf{j}\cosh 2t$. **1121.** 0. **1122.** $3(t^2 - 2t^5)\mathbf{i} + (5t^4 - 2t)\mathbf{j}$. **1124.** $x/(-1) = (y-1)/0 = (z - \pi\sqrt{3}/2)\sqrt{3}$, $2x - 2z\sqrt{3} + 3\pi = 0$. **1125.** $M_1(0; 0; -1)$ and $M_2(2/3; -8/9, -1/27)$. **1126.** $70°23'$. **1127.** $x/1 = (y-1)/0 = (z - \sqrt{2}/2)/1$, $x + z - \sqrt{2}/2 = 0$. **1128.** $(x-1)/2 = (y+1)/0 = (z-1)/1$. **1129.** $(x-1)/2 = (y-1)/3 = (z-1)/4$. **1130.** $\arccos(14/(3\sqrt{29}))$. **1132.** $ds = \sqrt{a^2 + b^2}dt$. **1133.** $h = 2\sqrt{3}\pi$. **1134.** $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = -3\mathbf{i}\sin t + 3\mathbf{j}\cos t + 4\mathbf{k}$, $\mathbf{w} = \dfrac{d^2\mathbf{r}}{dt^2} = -3\mathbf{i}\cos t - 3\mathbf{j}\sin t$.

1135. $\mathbf{v}|_{t=1} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{w}|_{t=1} = 2\mathbf{j} + 6\mathbf{k}$. **1137.** $\tau = (5/13)\mathbf{i} - (12/13)\mathbf{j}\sin t + (12/13)\mathbf{k}\cos t$. **1138.** $\tau = -(1/3)\mathbf{j} + (2\sqrt{2}/3)\mathbf{k}$. **1147.** $(2/3)\mathbf{i} + (2/3)\mathbf{j} + (1/3)\mathbf{k}$. **1148.** $(1/3)\mathbf{i} - (2/3)\mathbf{j} + (2/3)\mathbf{k}$. **1149.** $(-2/3)\mathbf{i} + (1/3)\mathbf{j} + (2/3)\mathbf{k}$. **1150.** 2/27. **1151.** 2/27. **1152.** $X - 2Y + 2Z - 2 = 0$. **1153.** $2X - Y - 2Z - 7 = 0$. **1154.** $2X + 2Y + Z - 19 = 0$.

Chapter 8

1160. $x^2 + y^2 \geqslant 1$, a portion of the plane outside the unit circle with centre at the origin. **1161.** A portion of the plane inside the circle $x^2 + y^2 < 1$. **1162.** A band between the parallel lines $x + y \leqslant 1$ and $x + y \geqslant -1$. **1163.** Concentric rings $\pi/2 \geqslant x^2 + y^2 \geqslant 0$, $5\pi/2 \geqslant x^2 + y^2 \geqslant 3\pi/2, \ldots$. **1164.** $y > x$, a half-plane lying above the bisector $y = x$. **1165.** A half-plane $x \geqslant 0$. **1166.** A sphere $x^2 + y^2 + z^2 \leqslant a^2$. **1167.** A portion of the space outside the cone $x^2 + y^2 - z^2 = 0$. **1168.** A portion of the space inside the sphere $x^2 + y^2 + z^2 < 1$, except for the origin. **1169.** A portion of the space above the plane $x + y + z = 0$, the plane inclusive. **1170.** A family of parallel straight lines $2x + y = C$. **1171.** A family of straight lines $y = Cx$. **1172.** A family of straight lines $y = e^{2C}x$, or $y = C_1x$ $(C > 0)$. **1173.** A family of parabolas $y = C\sqrt{x}$. **1174.** A family of equilateral hyperbolas $xy = C$ (for $C \neq 0$); a collection of the

coordinate axes Ox and Oy (for $C = 0$). **1175.** A family of planes $x + y + 3z = C$. **1176.** A family of spheres $x^2+y^2+z^2=C$. **1177.** A family of two-sheet hyperboloids $x^2-y^2-z^2 = C$ (for $C>0$); a family of one-sheet hyperboloids $x^2-y^2-z^2=C$ (for $C<0$); a cone $x^2-y^2-z^2=0$ (for $C = 0$). **1183.** $\frac{\partial u}{\partial x} = 2x - 3y - 4$, $\frac{\partial u}{\partial y} = 4y - 3x + 2$. **1184.** $\frac{\partial r}{\partial \rho} = 2\rho\sin^4\theta$ $\frac{\partial r}{\partial \theta} = 4\rho^2\sin^3\theta\cos\theta$. **1185.** $\frac{\partial u}{\partial x} = \frac{2x}{y^2} - \frac{1}{y}$, $\frac{\partial u}{\partial y} = \frac{x}{y^2} - \frac{2x^2}{y^3}$.

1186. $\frac{\partial z}{\partial x} = e^{xy(x^2+y^2)}(3x^2y+y^3)$, $\frac{\partial z}{\partial y} = e^{xy(x^2+y^2)}(x^3+3y^2)$.

1187. $\frac{\partial u}{\partial x} = \frac{y}{\sqrt{x}}$, $\frac{\partial u}{\partial y} = 2\sqrt{x} + 6y^3\sqrt{z^2}$; $\frac{\partial u}{\partial z} = \frac{2y^2}{\sqrt[3]{z}}$.

1188. $\frac{\partial u}{\partial x} = \frac{1}{y}e^{x/y}$; $\frac{\partial u}{\partial y} = -\frac{x}{y^2}e^{x/y} + \frac{z}{y^2}e^{-z/y}$, $\frac{\partial u}{\partial z} = -\frac{1}{y}e^{-z/y}$.

1189. $\frac{\partial z}{\partial x} = -\frac{2xy}{(1+x^2)^2+y^2}$, $\frac{\partial z}{\partial y} = \frac{1+x^2}{(1+x^2)^2+y^2}$.

1190. $\frac{\partial z}{\partial x} = 6x^2(x^3+y^2)e^{(x^3+y^2)^2}$, $\frac{\partial z}{\partial y} = 4y(x^3+y^2)e^{(x^3+y^2)^2}$.

1191. $\frac{\partial u}{\partial x} = (y-z)(2x-y-z)$, $\frac{\partial u}{\partial y} = (x-z)(x-2y+z)$, $\frac{\partial u}{\partial z} = (x-y)(-y+2z-x)$.

1192. $\frac{\partial u}{\partial x} = (6x-y)e^{3x^2+2y^2-xy}$, $\frac{\partial u}{\partial y} = (4y-x)e^{3x^2+2y^2-xy}$.

1193. $\frac{\partial u}{\partial y} = xze^{xyz}\sin\frac{y}{x} + \frac{1}{x}e^{xyz}\cos\frac{y}{x}$. **1195.** ρ. **1200.** $dz = \frac{2(xdx+ydy)}{x^2+y^2}$.

1201. $dz = \frac{2(x\,dy - y\,dx)}{x^2+\sin(2y/x)}$. **1202.** $2(x\,dx+y\,dy)\cos(x^2+y^2)$. **1203.** $dz = xy\left(\frac{y}{x}dx + \ln x\,dy\right)$.

1204. $du = -\frac{1}{\sqrt{x^2+y^2}}\left(dx + \frac{ydy}{x+\sqrt{x^2+y^2}}\right)$.

1205. $dz = e^x[(x\cos y - \sin y)dy + (\sin y + \cos y + x\sin y)dx]$. **1206.** $dz = e^{x+y}[[(x+1)\cos y + y(\sin x + \cos x)]dx + [x(\cos y - \sin y) + (y+1)\sin x]dy]$. **1207.** $dz = \frac{2dx}{x^2-4} + \frac{2\cos y dy}{\sin^2 y + 4}$. **1208.** $du = e^{xyz}(yzdx + xzdy + xydz)$. **1209.** 1.08. **1210.** -0.03. **1211.** 1.013. **1212.** 3.037. **1213.** 1.05. **1218.** $6(x+y)$. **1219.** $-\sin(x+y)$. **1210.** $-4\cos(2x+2y)/\sin^2(2x+2y)$. **1221.** 0. **1222.** $x(x+2y)/(x+y)^2$. **1223.** $y(2-y^2)\cos xy - xy^2\sin xy$. **1224.** $\sin y\cos(x+\cos y)$. **1225.** $\frac{x^2-y^2}{(x^2+y^2)^2}[(dy)^2-(dx)^2] - \frac{4xydxdy}{(x^2+y^2)^2}$. **1226.** $-\cos(x+y)(dx+dy)^2$.

1227. $ae^y[e^y\sin(ax+e^y) - \cos(ax+e^y)]$. **1228.** 4. **1230.** $2[(dx)^2 - dxdy + (dy^2)]$. **1235.**

$4/\sin 2x$. **1236.** $2x(3x+2)/(x^2+3x+1)^2$. **1237.** $\dfrac{\partial z}{\partial x} = 2x\cos x, \dfrac{dz}{dx} = x(2\cos x - x\sin x)$.

1238. 0. **1239.** $\dfrac{\partial z}{\partial \xi} = 4\xi, \dfrac{\partial z}{\partial \eta} = 4\eta$. **1240.** $\dfrac{\partial u}{\partial \xi} = \dfrac{2}{\xi}, \dfrac{\partial u}{\partial \eta} = \dfrac{2(\eta^4-1)}{\eta(\eta^4+1)}$.

1246. $\left(\dfrac{\partial z}{\partial l}\right)_M = \dfrac{7}{5}$. **1247.** $\left(\dfrac{\partial u}{\partial l}\right)_M = \dfrac{1}{6}$.

1248. $\left(\dfrac{\partial u}{\partial l}\right)_M = \dfrac{7}{9}$. **1249.** $|\operatorname{grad} u|_M = 1/r_0^2$; $\cos\alpha = -x_0/r_0$, $\cos\beta = -y_0/r_0$, $\cos\gamma = -z_0/r_0$, where $r_0 = \sqrt{x_0^2+y_0^2+z_0^2}$.
1250. $|\operatorname{grad} u|_M = 3$, $\cos\alpha = 1/3$, $\cos\beta = \cos\gamma = 2/3$. **1251.** $1/3$. **1256.** $-x/y$. **1257.** y/x. **1258.** $y' = -y/x, y'' = 2y/x^2$ **1259.** $(y\sqrt{2xy} - x^2)/(2y^2 - x\sqrt{2xy})$. **1260.** y/x. **1261.** $(a^2 - + b^2)/(2b^2 - a^2)$. **1262.** $y/(2x)$. **1263.** $(x+y)/(x-y)$. **1264.** $1/(2^y \ln 2)$.

1265. $y' = -1, y'' = 0$, **1266.** $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y} = \dfrac{1}{x+y+z-1}$. **1267.** $\dfrac{\partial z}{\partial x} = -\dfrac{x^2-yz}{z^2-xy}, \dfrac{\partial z}{\partial y} = -\dfrac{y^2-xz}{z^2-xy}$. **1268.** $\dfrac{dx+(z/y)dy}{1+\ln(z/y)}$. **1269.** $-\dfrac{x\cos y+\sin x}{\sin x}$. **1270.** $-\dfrac{(y+z)dx+(x+z)dy}{x+y}$.

1271. $-y-z-e^{y-x}$. **1272.** 1. **1275.** $2x+2y-z=1$, $(x-1)/2=(y-1)/2=(z-3)/(-1)$. **1276.** $2x+2y-3z+1=0$, $(x-2)/2=(y-2)/2=(z-3)/(-3)$. **1277.** $z-2x+2=0$, $(x-1)/2=y/0=z/(-1)$. **1278.** $x-y-2z+1=0$, $(x-\pi/4)/1 = (y-\pi/4)/(-1) = = (z-1/2)/(-2)$. **1279.** $x+4y+6z\pm 21=0$. **1281.** $(4/3; 4/3; 1/3)$ and $(-4/3; -4/3; -1/3)$. **1285.** $z_{max} = 1/64$. **1286.** $z_{min} = -125$. **1287.** $z_{max} = 4$. **1288.** $z_{min} = 0$. **1289.** $z_{max} = a\sqrt{3}/9$ at $x=y=2a/3$. **1293.** $z_{min} = 144/25$ at the point $(36/25; 48/25)$. **1294.** $z_{least} = -16/3$, $z_{gr} = 16$. **1295.** $z_{least} = 5$, $z_{gr} = 11$. **1296.** $z_{least} = -1/2$, $z_{gr} = 1/2$. **1297.** $z_{least} = 1$, $z_{gr} = 4$. **1298.** $z_{least} = = -2(\sqrt{2}+1) \cong -4.8$, $z_{gr} = 2(\sqrt{2}-1) \cong 0.8$. **1299.** $z_{least} = 0$, $z_{gr} = 3\sqrt{3}/2$. **1300.** $z_{least} = = -3$ for $x=y=3\pi/2$, $z_{gr} = 1+\sqrt{3}/2$ for $x=y=5\pi/6$. **1301.** $z_{least} = -1/8$, $z_{gr} = 1$. **1302.** Equilateral. **1303.** Equilateral. **1304.** A square; $P_{least} = 4\sqrt{S}$. **1305.** A cube; $V_{max} = (S/6)\sqrt{S/6}$.

Chapter 9

1315. $(2/5)x^2\sqrt{x} + C$. **1316.** $(5/4)\sqrt[5]{x^4} + C$. **1317.** $2\arcsin x - x + C$. **1318.** $\arctan x + x - - x^3/3 + C$. **1319.** $e^{3x}\cdot 3^x/(3+\ln 3) + C$. **1320.** $\tan x - x + C$. **1321.** $\cosh x + \cos x + C$. **1322.** $x^2/2 - 2x + \ln|x| + C$. **1323.** $4\tan x - 9\cot x - x + C$. **1324.** $(1/2)\sin(x^2) + C$. **1325.** $\ln|\ln x| + C$. **1326.** $3(ax^2+b)^{4/3}/(8a) + C$. **1327.** $(2/3)\sin x\sqrt{\sin x} + C$. **1328.** $-(1/b)\cos(a+ + bx) + C$. **1329.** $\sin(\sin x) + C$. **1346.** $e^{\sqrt{2x-1}} + C$. **1347.** $-(1/32)(1-2x^4)^4 + C$. **1348.** $(1/3)\cos(2-3x) + C$. **1349.** $(1/10)\sinh(5x^2+3) + C$. **1350.** $2\arctan\sqrt{x} + C$. **1351.** $(1/5)(x^2+ + 1)^{5/2} + C$, **1352.** $(1/2)\ln|x^2-1| + C$. **1353.** $(1/2)\ln|x^2+\sqrt{x^4-1}| + C$. **1354.** $(-1/4)\arctan(0.5\cos^2 2x) + C$. **1355.** $(1/\sqrt{7})\ln|(\sqrt{x}-\sqrt{7})/(\sqrt{x}+\sqrt{7})| + C$. **1356.** $2\arcsin(e^{x/2}/4) + C$. **1357.** $\ln|x+\sqrt{2+x^2}| + \arcsin(x/\sqrt{2}) + C$. **1358.** $(-2/9)\sqrt{2-3x^3} + C$. **1359.** $-5\sqrt{3-x^2} + 3\arcsin(x/\sqrt{3}) + C$. **1360.** $(1/4)\arctan(x-3)/4 + C$. **1361.** $2\sqrt{3x+5} + + \sqrt{5}\ln|(\sqrt{3x+5}-\sqrt{5})/(\sqrt{3x+5}+\sqrt{5})| + C$. **1362.** $1/(2\sqrt{10})\arctan(x^2\sqrt{2/5}) + C$. **1370.** $(x^2/4)(2\ln x - 1) + C$. **1371.** $x\arcsin x + \sqrt{1-x^2} + C$. **1372.** $(x^2/3)\arctan x - (1/6)x^2 +$

$+ (1/6)\ln(x^2 + 1) + C$. **1373.** $xe^x + C$. **1374.** $- x^2\cos x + 2x\sin x + 2\cos x + C$. **1375.** $(1/2)e^{x^2}(x^4 - 2x^2 + 2) + C$. **1376.** $(x + 1)^2\sin x + 2(x + 1)\cos x + C$ **1377.** $(e^{2x}/5)(\sin x + 2\cos x) + C$. **1378.** $(x/2)(\sin\ln x - \cos\ln x) + C$. **1379.** $- 2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$. **1387.** $- 1/[3(x - 1)^3] + C$. **1388.** $- 1/[4(2x + 3)^2] + C$. **1389.** $(1/3)\arctan(x - 3)/3 + C$. **1390.** $1/(3\sqrt{2})\arctan(x^3 + 1)/\sqrt{2} + C$. **1391.** $(1/2)\ln(x^2 - 4x + 7) + C$. **1392.** $(5/2)\ln(x^2 + 10x + 29) - 11\arctan(x + 5)/2 + C$. **1393.** $(1/10)\ln(5x^2 + 2x + 1) + (2/5)\arctan(5x + 1)/2 + C$. **1394.** $x/[8(x^2 + 2)^2] + 3x/[32(x^2 + 1)] + (3\sqrt{2}/64)\arctan(x/\sqrt{2}) + C$. **1395.** $(x - 7)/[8(x^2 + 2x + 5)] + (1/16)\arctan(x + 1)/2 + C$. **1405.** $- (2/3)\ln|x| + (5/3)\ln|x - 3| + C$. **1406.** $- (1/2)\ln(x^2 + x + 1) + 3\ln|x + 2| + (1/\sqrt{3})\arctan(2x + 1)/\sqrt{3} + C$. **1407.** $1/[2(x - 1)^2] + 2\ln|x - 1| + 3\ln|x - 2| + C$. **1408.** $(1/12)\ln|x - 2| - (1/24)\ln(x^2 + 2x + 4) - 1/(4\sqrt{3})\arctan(x + 1)/\sqrt{3} + C$. **1409.** $(31/108)\ln|x - 3| + (29/108)\ln|x + 3| + (2/9)\ln(x^2 + 9) - (1/54)\arctan(x/3) + C$. **1410.** $(1/4)\ln|x/(x-2)| - (x-1)/[2x(x-2)] + C$. **1411.** $(1/16)[\ln(x^2 + 1)/(x^2 + 9)] + (1/8)\arctan(x/2) - (1/24)\arctan(x/3) + C$. **1412.** $(1/4)\ln[(x^2 + 4)/(x^2 + 2x + 5)] + (1/8)\arctan(x/2) + (7/32)\arctan(x + 1)/2 + C$. **1413.** $(1/2)[\arctan(x - 1) + \arctan(x + 1)] + C$. **1414.** $x + (9/2)\ln|x - 3| - (1/2)\ln|x - 1| + C$. **1415.** $x^2/2 + 7x + (75/2)\ln|x - 5| - (1/2)\ln|x - 1| + C$. **1416.** $x + (1/2)\ln|(x - 2)/(x + 2)| - \arctan(x/2) + C$. **1417.** $3x + \ln|x| + 2\arctan x + C$. **1427.** $-\sqrt{1 - 2x} - 2\sqrt[4]{1 - 2x} - 2\ln \times \times |\sqrt[4]{1 - 2x} - 1| + C$. **1428.** $(6/5)\sqrt[6]{x^5} - 2\sqrt{x} + 6\sqrt[6]{x} - 6\arctan\sqrt[6]{x} + C$. **1429.** $\ln|x - 1/2 + \sqrt{x^2 - x - 1}| + C$. **1430.** $\arcsin(x + 1)/3 + C$. **1431.** $-5\sqrt{-x^2 + 4x + 5} + 13\arcsin(x - 2)/3 + C$. **1432.** $3\sqrt{x^2 + x + 2} + (1/2)\ln|x + 1/2 + \sqrt{x^2 + x + 2}| + C$. **1433.** $\sqrt{x/(x + 2)} + C$. **1434.** $-\arcsin[(x + 1)/(x\sqrt{3})] + C$. **1435.** $\ln|x + \sqrt{x^2 + 1}| + \sqrt{2}\ln\left|\dfrac{1 - x + \sqrt{2(x^2 + 1}}{2(x + 1)}\right| + C$. **1436.** $-(x/2 + 5\times\sqrt{-x^2 + 4x} + 13\arcsin(x - 2)/2 + C$. **1440.** $3/\sqrt[3]{x + 1} + \ln[x/(\sqrt[3]{x} + 1)^3] + C$. **1441.** $4\sqrt{\sqrt{x} + 1}[(1/5)(\sqrt{x} + 1)^2 - (2/3)(\sqrt{x} + 1) + 1] + C$. **1442.** $(1/6)\ln[(t^2 + t + 1)/(t^2 - 2t + 1)] - (1/\sqrt{3})\arctan(2t + 1)/\sqrt{3} + C$, where $t = \sqrt[3]{1 + x^3}/x$. **1443.** $(1/3)\ln[(\sqrt{1 + x^3} - 1)^2/|x^3|] + C$. **1444.** $(1/10)(5x^{4/3} + 3)^{3/2} + C$. **1445.** $- (2 - x^3)^{2/3}/(4x^2) + C$. **1463.** $(1/5)\ln|5\tan(x/2) + 3| + C$. **1464.** $- 2/[\tan(x/2) - 1] + C$. **1465.** $- (1/17)x + (1/4)\ln|\sin x| - (1/68)\ln|\sin x + 4\cos x| + C$. **1466.** $\ln|\sin x| - \sin x + C$. **1467.** $- (2/5)\ln(1 - \cos x) + (1/5)\ln(\cos^2 x + 2\cos x + 2) - (6/5)\arctan(1 + \cos x) + C$. **1468.** $(1/3)\cos^3 x - \cos x + C$. **1469.** $\ln|\sin x| - \sin^2 x + (1/4)\sin^4 x + C$. **1470.** $(1/8)x - (1/8)\sin x + C$. **1471.** $(3/8)x + (1/4)\sin 2x + (1/32)\sin 4x + C$. **1472.** $(2/3)\tan^3(x/2) - 2\tan(x/2) + x + C$. **1473.** $-(1/6)\cot^2 3x - (1/3)\ln|\sin 3x| + C$. **1474.** $\tan x + (2/3)\tan^3 x + (1/5)\tan^5 x + C$. **1475.** $-(1/3)\cot^3 x + C$. **1476.** $(1/2)\tan x\sec x + (1/2)\ln|\tan(x/2 + \pi/4)| + C$. **1477.** $- (1/2)\cot x\operatorname{cosec} x - (1/2)\ln|\tan(x/2)| + C$. **1478.** $(1/4)\sin 2x - (1/8)\sin 4x + C$. **1479.** $(3/5)\sin(5x/6) + 3\sin(x/6) + C$. **1483.** $x/\sqrt{1 - x^2} + C$. **1484.** $x/(a^2\sqrt{a^2 + x^2}) + C$. **1485.** $(1/2)(\arccos(1/x) + \sqrt{x^2 - 1}/x^2) + C$. **1486.** $- (1/6)\cos 3x + (1/20)\cos 5x + (1/4) \times \times \cos x + C$. **1487.** $-2(2x^2 + 6x + 13)e^{-x/2} + C$. **1488.** $-(2\ln x + 1)/(4x^2) + C$. **1489.** $x\arctan x - (3/2)(\arctan x)^2 - (1/2)\ln(1 + x^2) + C$. **1490.** $(x^2 - 1)\sin 2x + x\cos 2x + C$. **1491.** $(1/4)x^2(2\ln^2 x - 2\ln x + 1) + C$. **1492.** $x + \ln|e^x - 3| + C$. **1493.** $(x + 1)\arctan\sqrt{x} - \sqrt{x} + C$. **1494.** $(2/\ln 2)(\sqrt{2^x - 1} - \arctan\sqrt{2^x - 1}) + C$. **1495.** $(1/4)\ln|(t + 1)/(t - 1)| - (1/2) \times \times \arctan t + C$, where $t^4 = 1 + x^{-4}$. **1496.** $\sqrt{2}[(1/2)(x - 1)\sqrt{3 + 2x - x^2} + 2\arcsin(x - 1)/2] + C$. **1497.** $\sin e^x - e^x\cos e^x + C$. **1498.** $(1/\sqrt{5})\ln|\tan x + \sqrt{\tan^2 x + 2/5}| + C$. **1499.** $(1/2) \times \times \cos^2 x(1 - 2\ln\cos x) + C$. **1500.** $(1/2)\sin(x^2 + 4x + 1) + C$. **1501.** $-0.5(x/\sin^2 x + \cot x) + C$.

1502. $2(x-2)\sqrt{1+e^x} - 2\ln[(\sqrt{1+e^x}-1)/(\sqrt{1+e^x}+1)] + C$. **1503.** $x\ln(x^2+x) + \ln|x+1| - x + C$. **1504.** $-1/x - \arctan x + C$. **1505.** $(x/2)(\cos\ln x + \sin\ln x) + C$. **1506.** $12[\sqrt[12]{x} + \ln[(\sqrt[12]{x}-1)^2/\sqrt[12]{x}]] + C$. **1507.** $[e^{\alpha x}/(\alpha^2+\beta^2)] \times (\alpha\sin\beta x - \beta\cos\beta x) + C$. **1508.** $[e^{\alpha x}/\alpha^2+\beta^2)](\beta\sin\beta x + \alpha\cos\beta x) + C$. **1509.** $[1/(ab)]\arctan[(b/a)\tan x] + C$. **1510.** $\tan x - \cot x + C$. **1511.** $0.5(\arctan x + x/(1+x^2)) + C$.

Chapter 10

1521. $1/2$. **1522.** $e-1$. **1523.** $0 < I \leqslant 4/27$. **1524.** $\pi/2 \leqslant I \leqslant e\pi/2$. **1525.** $0 < I < 1$. **1526.** $464\sqrt{2}/15$. **1527.** $\pi/8$. **1528.** $e - \sqrt{e}$. **1529.** $e^e - e$. **1530.** $(e^{\pi/2}-1)/2$. **1531.** $(\ln 3 - 1)/2$. **1532.** $\ln 1.5$. **1533.** 0. **1534.** $2/5$. **1535.** $\pi/2$. **1536.** $\ln(4/3)$. **1537.** $(e^{\pi/2}-1)/2$. **1538.** 0. **1539.** $\pi/2 - 1$. **1546.** $\pi^2/8$. **1547.** $\pi/4$. **1548.** $256/15$. **1549.** π. **1550.** $+\infty$. **1551.** $1/4$. **1552.** $\pi/6$. **1559.** Diverges. **1560.** Converges. **1561.** Diverges. **1562.** Converges. **1563.** Diverges. **1564.** Converges. **1565.** Diverges. **1569.** 4.5 (sq. units). **1570.** 18 (sq. units). **1571.** 2/15 (sq. units). **1572.** $(41/2)\arcsin(9/41) + 20\ln 0.8$ (sq. units). **1573.** $\sqrt{2}-1$ (sq. units). **1574.** 8 (sq. units). **1575.** $(9\pi/4) - \sqrt{2} + 4\sqrt{2}\ln 2 - (9/2)\arcsin(1/3)$ (sq. units). **1576.** 169π (sq. units). **1577.** $(3/8)\pi a^2$. **1578.** $8\sqrt{3}/3$ (sq. units). **1579.** $(3/2)\pi a^2$. **1580.** $(3\pi-8)/32$ (sq. units). **1581.** $\pi a^2/12$. **1582.** $\pi/3 + \sqrt{3}/2$ (sq. units). **1586.** $(1/2)\ln 3$. **1587.** $(20/9)\sqrt{5/3}$. **1588.** $0.5[\sqrt{2} + \ln(1+\sqrt{2})]$. **1589.** $(1/2)\ln 3$. **1590.** $\sinh 1 \approx 1.17$. **1591.** 12. **1592.** $\sqrt{2}(\pi-1)$. **1593.** 5π. **1594.** 72. **1595.** $[(\pi^2+4)\sqrt{\pi^2+4} - 8]/3$. **1596.** πa. **1597.** $a(2\pi + 3\sqrt{3})/8$. **1598.** 8. **1601.** $16\pi(5\pi+8)/5$ (cu. units). **1602.** 0.3π (cu. units). **1603.** $\pi(e^2+1)/4$ (cu. units). **1604.** $4\pi/35$ (cu. units). **1605.** 72 (cu. units). **1606.** $2a^2h/3$. **1607.** $\pi r^2h/2$. **1609.** $\pi(e^2 - e^{-2} + 4)$ (sq. units). **1610.** $61\pi/1728$ (sq. units). **1611.** $2\pi b\,[b + (a^2/c^2)\arcsin(c/a)]$, where $c^2 = a^2 - b^2$. **1612.** $64\pi/3$ (sq. units). **1617.** $M_x = a^2(e^2 - e^{-2} + 4)/8$; $I_x = a^3(e - e^{-1})(e^2 + e^{-2} + 10)/24$. **1618.** $M_a = ah^2/6$; $I_a = ah^3/12$. **1619.** $I_x = 1628/105$. **1620.** $I_x = ab^3/12$; $I_y = a^3b/12$. **1621.** $I_0 = \pi d^4/32$. **1626.** $\bar{x} = 0, \bar{y} = 2r/\pi$ (for a semicircle); $\bar{x} = 0, \bar{y} = 4r/(3\pi)$ (for half a disc). **1627.** $\bar{x} = (\pi-2)/2, \bar{y} = \pi/8$. **1628.** $\bar{x} = 0, \bar{y} = 8/5$. **1629.** $\bar{x} = \bar{y} = 2a/5$. **1630.** $\bar{x} = 1, \bar{y} = 2/5$. **1631.** $\bar{x} = 1, \bar{y} = \pi/8$. **1632.** $\pi r^3(3\pi-4)/3$. **1634.** 480π (cu. units). **1643.** $\pi\rho g r^2h^2/4$. **1644.** $\pi\rho g a d^3/8$. **1645.** $51\,450\pi$ J. **1646.** 547.8π J. **1647.** 50.7 J. **1648.** $\rho g a h^2/3$. **1649.** 17.64π kPa; 70.56π kPa; 158.76π kPa; 282.24π kPa. **1650.** $\rho g\pi d^3/8$. **1651.** 150 kg. **1652.** 1400 m. **1653.** $x = e^{10}$. **1654.** 36 m. **1663.** (1) $y' = \sinh x/\sqrt{\cosh^2 x + 1}$; (2) $y' = \sinh^2(x/15)\cosh^3(x/15)$; (3) $y' = 1/\cosh x$; (4) $y' = 1/\cosh^6 x$; (5) $y' = 1/\cosh x$; (6) $y' = x/\sinh(x^2/2)$. **1664.** $M[\ln(1+\sqrt{2}); \sqrt{2}]$; **1665.** $y_{\min} = 0$ at $x = 0$. **1666.** (1) $x^2\sinh x - 2x\cosh x + 2\sinh x + C$; (2) $(1/32)\sinh 4x - (1/4)\sinh 2x + 6x + C$; (3) $2\arctan\sqrt{\cosh x - 1} + C$; (4) $(1/2)(\cosh x\sinh x - \sin x\cos x) + C$; (5) $\tanh^2(x/2) + C$; (6) $(3/5)\cosh^5(x/3) - \cosh^3(x/3) + C$. **1667.** (1) $\pi/6$; (2) $\ln 2 - 0.6$; (3) $2(2\ln 3 - 1)/3$. **1668.** $\sinh(x-a) = \sinh x\cosh a - \cosh x\sinh a$; $\cosh(x-a) = \cosh x\cosh a - \sinh x\sinh a$. **1669.** $\tanh(x+a) = (\tanh x + \tanh a)/(1 + \tanh x\tanh a)$; $\tanh(x-a) = (\tanh x - \tanh a)/(1 - \tanh x\tanh a)$; $\tanh 2x = 2\tanh x/(1+\tanh^2 x)$. **1670.** $\sinh(x/2) = \pm\sqrt{(\cosh x - 1)/2}$; $\cosh(x/2) = \sqrt{(\cosh x + 1)/2}$; $\tanh(x/2) = \pm\sqrt{(\cosh x - 1)/(\cosh x + 1)}$. **1671.** $2\sinh(x \pm y)/2 \cdot \cosh(x \mp y)/2$; $2\cosh(x+y)/2 \cdot \cosh(x-y)/2$; $2\sinh(x+y)/2 \cdot \sin(x-y)/2$; $\sinh(x \pm$

$\pm\, y)/(\cosh x \cosh y)$. **1672.** $\sinh x = 2 \tanh (x/2)/[1 - \tanh^2 (x/2)]$; $\cosh x = [1 + \tanh^2 (x/2)]/[1 - \tanh^2 (x/2)]$. **1673.** $(1/2) \cdot [\sinh (x + y) + \sinh (x - y)]$; $(1/2) \cdot [\cosh (x + y) + \cosh (x - y)]$; $(1/2) \cdot [\cosh (x + y) - \cosh (x - y)]$. **1674.** 8/5 (sq. units). **1675.** 1.17a. **1676.** $a^2 (t_2 - t_1)/2$. **1677.** An arc of an allipse located above the abscissa axis. **1678.** A ray of the straight line $x - y - 1 = 0$ located in the first quadrant. **1679.** $\cos \alpha = \pm 1/\cosh t$; $\tan \alpha = \pm \sinh t$. **1680.** 1. **1681.** $x^2 - y^2$.

Chapter 11

1692. Below the straight line $x_1 - x_2 - 10 = 0$. **1693.** A triangle. **1694.** An unbounded domain. **1695.** An empty domain. **1696.** Point (2; 3). **1697.** The domain of solutions is a trapezoid, inequality (e) can be eliminated. **1698.** A triangular pyramid. **1699.** A triangular prism. **1704.** $L_{max} = 10$ at $x_1 = 4, x_2 = 2$. **1705.** $L_{min} = -4$ at $x_1 = 0, x_2 = 4$. **1706.** $L_{max} = 18$ at $x_1 = 6, x_2 = 0$. **1707.** $L_{max} = 2$ at $x_1 = 0, x_2 = 1$. **1708.** $L_{max} = 48$ at every point of the line segment AB, where $A\,(6;\ 0)$, $B(7;\ 4)$. **1709.** $L_{min} = -4$ at $x_1 = 2, x_2 = 0, x_3 = 0$. **1710.** $L_{max} = 33$ at $x_1 = 0, x_2 = 3, x_3 = 9$. **1711.** $L_{max} = 20$ at $x_1 = 2, x_2 = 0, x_3 = 0$. **1718.** (0, 1, 3, 0); $L = -3$. **1719.** (3/2, 0, 0, 1/2, 11/2); $L = 3/2$. **1720.** (0, 3, 1, 0), $L = -1$. **1721.** (0,50, 30, 215/6); $L = 340$. **1722.** (0, 5, 3/2, 0, 0, 3/2); $L = -5$. **1723.** $L = \infty$. **1724.** (0, 4/5, 1/5); $L = -11/5$. **1725.** (80, 120); $L_{max} = 440$. **1726.** (20, 40); $L_{max} = 220$. **1727.** 15 kg of forage of the first kind; 50 kg of forage of the second kind. **1729.** (5, 7, 9, 7/2, 0, 0); $L_{max} = 38$. **1731.** $L_{min} = 86/9$; (10/9, 28/9); $T_{max} = 86/9$; (7/27, 8/27). **1732.** $L_{max} = 216/7$; (24/7, 24/7); $T_{min} = 216/7$; (8/7, 1/7). **1733.** $L_{min} = 78/7$; (10/7, 16/7); $T_{max} = 78/7$; (27/14, 3/14). **1735.** $L_{min} = 510$ rubles; $x_{11} = 30, x_{12} = 60, x_{21} = 30, x_{23} = 60$. **1736.** $L_{min} = 280$ rubles. The optimal plan is $x_{12} = x_{24} = x_{33} = 60, x_{23} = 20, x_{31} = 40$. **1737.** $L_{min} = 395$ rubles. The optimal plan is $x_{11} = 25, x_{13} = x_{32} = 20, x_{12} = x_{21} = 5, x_{23} = 35$. **1738.** $L_{min} = 140$ rubles. The optimal plan is $x_{14} = x_{22} = 10, x_{23} = x_{31} = x_{34} = 5, x_{33} = 15$.

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